Antenna-Independent Capacity Bound of Electromagnetic Channels

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1 Introduction

Studies of wireless multiple-input multiple-output (MIMO) channel capacity generally treat the channel as a propagation environment coupled with an antenna array. However, it is well known that altering antenna properties can have a dramatic impact on the MIMO performance. This motivates definition of a capacity bound that depends only on the propagation environment and is independent of the antenna geometry. This paper provides a framework for formulating this capacity, and demonstrates its behavior for several representative scenarios.

2 Traditional Channel Description

We will use boldface uppercase and lowercase letters to describe matrices and column vectors, respectively. A dyadic Green's function $\mathbf{G}_{\mathrm{rt}}$ relates the vector transmit current distribution $\mathbf{j}(\mathbf{r}_{\mathrm{t}})$ to the received electromagnetic fields according to

$$\mathbf{e}_{\mathrm{r}}(\mathbf{r}_{\mathrm{r}}) = \int_{\Delta V_{\mathrm{t}}} d\mathbf{r}'_{\mathrm{t}}\ \mathbf{G}_{\mathrm{rt}}(\mathbf{r}_{\mathrm{r}}, \mathbf{r}'_{\mathrm{t}})\mathbf{j}(\mathbf{r}'_{\mathrm{t}}) + \mathbf{e}_{\eta}(\mathbf{r}_{\mathrm{r}}), \tag{1}$$

where $\mathbf{e}_{\eta}(\mathbf{r}_{\mathrm{r}})$ is additive noise. If we consider this integration as a compact operator acting on the currents, we can expand (1) using a basis set of orthonormal eigenfunctions $\boldsymbol{\rho}_m(\mathbf{r}_{\mathrm{r}})$ and $\boldsymbol{\tau}_m(\mathbf{r}_{\mathrm{t}})$ to obtain [1]

$$\begin{aligned} y_m &= x_m H_m + \eta_m, & (2)\\ H_m &= \int_{\Delta V_{\mathrm{r}}} d\mathbf{r}_{\mathrm{r}} \int_{\Delta V_{\mathrm{t}}} d\mathbf{r}'_{\mathrm{t}} \boldsymbol{\rho}^{\dagger}_m(\mathbf{r}_{\mathrm{r}})\mathbf{G}_{\mathrm{rt}}(\mathbf{r}_{\mathrm{r}}, \mathbf{r}'_{\mathrm{t}})\boldsymbol{\tau}_m(\mathbf{r}'_{\mathrm{t}}) & (3)\\ \eta_m &= \int_{\Delta V_{\mathrm{r}}} d\mathbf{r}_{\mathrm{r}} \boldsymbol{\rho}^{\dagger}_m(\mathbf{r}_{\mathrm{r}})\mathbf{e}_{\eta}(\mathbf{r}_{\mathrm{r}}), & (4) \end{aligned}$$

where H_m are channel eigenvalues. Typically, the channel eigenfunctions will span a finite-dimensional space. Even if these eigenfunctions span an infinite-dimensional space, for practical channel representations only a finite set of these eigenfunctions will lead to significant eigenvalues H_m that will be used in a water-filling capacity formulation. Therefore, we can formulate the capacity using matrix notation.

To formulate the capacity, we require a power constraint and a noise model. The continuous-space analog of the traditional power constraint in discrete-space MIMO capacity analysis can be manipulated under this basis expansion to

$$\int_{\Delta V_{\mathrm{t}}} d\mathbf{r}'_{\mathrm{t}}\ \mathrm{E}\left\{\left\|\mathbf{j}(\mathbf{r}'_{\mathrm{t}})\right\|_2^2\right\} = \sum_m \mathrm{E}\left\{|x_m|^2\right\} = \mathrm{Tr}(\mathbf{R}_x) \le P_T, \tag{5}$$

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where $\|\cdot\|_2$ denotes an L_2 vector norm, $\mathbf{R}_x = \mathrm{E}\left\{\mathbf{x}\mathbf{x}^\dagger\right\}$ and P_T is the maximum allowable transmit power. For thermal noise generated in the receiver, $\mathbf{e}_\eta(\mathbf{r}_\mathrm{r})$ is a spatially uncorrelated zero-mean complex Gaussian random process, which yields $R_{\eta,mp} = \mathrm{E}\left\{\eta_m \eta_p^*\right\} = \sigma_\eta^2 \delta_{mp}$. These assumptions lead to the capacity expression

$$C = \max_{\mathbf{R}_x} \log_2 \det\left[\mathbf{H}\mathbf{R}_x\mathbf{H}^\dagger/\sigma_\eta^2 + \mathbf{I}\right], \tag{6}$$

where $\mathbf{H}$ is a diagonal matrix containing the eigenvalues H_m. The optimal covariance $\mathbf{R}_x$ can be determined using the water-filling solution with (5).

The set of channels and transmit/receive volumes for which the eigenfunctions can be determined in closed form is limited. In this case we numerically approximate these eigenfunctions using *sub-basis functions* according to

$$\boldsymbol{\tau}_n(\mathbf{r}_\mathrm{t}) = \sum_q C_{qn}\tilde{\boldsymbol{\tau}}_q(\mathbf{r}_\mathrm{t}) \qquad \boldsymbol{\rho}_m(\mathbf{r}_\mathrm{r}) = \sum_p B_{pm}\tilde{\boldsymbol{\rho}}_p(\mathbf{r}_\mathrm{r}). \tag{7}$$

Generalizing (3) by replacing $\boldsymbol{\tau}_m$ with $\boldsymbol{\tau}_n$ and using (7) gives

$$H_{mn} = \int_{\Delta V_\mathrm{r}} d\mathbf{r}_\mathrm{r} \int_{\Delta V_\mathrm{t}} d\mathbf{r}'_\mathrm{t} \left[\sum_p B_{pm}\tilde{\boldsymbol{\rho}}_p(\mathbf{r}_\mathrm{r})\right]^\dagger \mathbf{G}_{rt}(\mathbf{r}_\mathrm{r},\mathbf{r}'_\mathrm{t}) \sum_q C_{qn}\tilde{\boldsymbol{\tau}}_q(\mathbf{r}'_\mathrm{t}) \tag{8}$$

or $\mathbf{H} = \mathbf{B}^\dagger\widetilde{\mathbf{H}}\mathbf{C}$. Representing $\widetilde{\mathbf{H}}$ in terms of its singular value decomposition (SVD) $\widetilde{\mathbf{H}} = \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^\dagger$, we assign $\mathbf{C} = \widetilde{\mathbf{V}}$ and $\mathbf{B} = \widetilde{\mathbf{U}}$ to obtain $\mathbf{H} = \widetilde{\mathbf{S}}$ which is diagonal as desired. Also, since $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\rho}}$ are orthonormal and $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ are unitary, this assignment ensures that $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are also orthonormal.

To illustrate application of this technique, we assume a single electromagnetic polarization with propagation confined to the horizontal plane and described by the model in [2]. We allow the identical transmit and receive regions to be infinite in the z direction with square cross-sectional areas characterized by side lengths $\Delta x = \Delta y$. The sub-basis functions are pulse functions with height $N/\sqrt{\Delta x \Delta y}$, where N is the number of subdivisions in x and y.

We use $\Delta x = \Delta y = 1\lambda$ and 2λ, where λ is the free-space wavelength, along with a single realization of the statistical path-based channel model consisting of 78 different paths. Using a single sub-basis function in the transmit and receive volumes ($N = 1$), we compute the channel gain $\widetilde{H}_{11}$ and define SNR $= P_T|\widetilde{H}_{11}|^2/\sigma_\eta^2$. The value of σ_η^2 is then chosen to obtain an SNR of 20 dB, and this value is held constant as the number of sub-basis functions increases. Fig. 1 shows the capacity obtained from this computation as a function of N for the two aperture dimensions. The capacity approaches an upper bound as the number of sub-basis functions increases.

3 Modified Channel Description

Consider now the case where we limit the power radiated by the transmit array and assume the noise is generated external to the receive array. The power radiated by the currents can be written as

$$p_\mathrm{rad} = \frac{1}{2Z_0}\int_{\Delta V_\mathrm{t}} d\mathbf{r}_{\mathrm{t}1} \int_{\Delta V_\mathrm{t}} d\mathbf{r}_{\mathrm{t}2}\mathbf{j}^\dagger(\mathbf{r}_{\mathrm{t}1}) \left[\oint d\Omega_\mathrm{t}\ \psi^*(\mathbf{r}_{\mathrm{t}1},\Omega_\mathrm{t})\psi(\mathbf{r}_{\mathrm{t}2},\Omega_\mathrm{t})\right] \mathbf{j}(\mathbf{r}_{\mathrm{t}2}) \tag{9}$$

where Z_0 is the free-space impedance and $\psi(\mathbf{r},\Omega) = e^{-jk(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta)}$. If we use the constraint $\mathrm{E}\left\{p_\mathrm{rad}\right\} \leq P_T$, we must find currents that do not waste

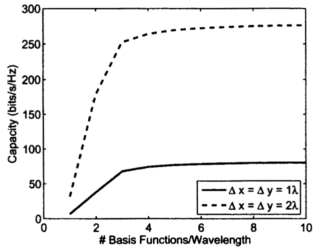

Figure 1: Numerically computed capacity versus N for the traditional channel model.

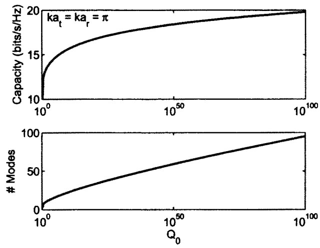

Figure 2: Capacity and number of eigenfunctions versus Q_0 for communication using circular cylinders.

power by radiating into the null space of the channel operator, and therefore must find a complete (infinite-dimensional) basis for representing the transmit current. Letting $\boldsymbol{\tau}_n(\mathbf{r}_\mathrm{t})$ be the eigenfunctions of the operation in (9) transforms this equation to $p_\mathrm{rad} = \sum_n x_n^* A_n x_n$ where A_n is obtained from (9) with $\mathbf{j}$ replaced by $\boldsymbol{\tau}$. Since the operator in (9) is self-adjoint, the eigenvalues A_n will be real. We can therefore define $\hat{x}_n = \sqrt{A_n} x_n$ so that $p_\mathrm{rad} = \sum_n |\hat{x}_n|^2$.

For noise generated external to the receive sub-system, we assume that a zero-mean complex Gaussian noise field $\mathbf{n}(\Omega_\mathrm{r})$ with $\mathrm{E}\left\{\mathbf{n}(\Omega_\mathrm{r})\mathbf{n}^\dagger(\Omega_\mathrm{r}')\right\} = \sigma_\eta^2 \mathbf{I}\delta(\Omega_\mathrm{r} - \Omega_\mathrm{r}')$ impinges on the receive volume. With this model, the received noise in (4) has cross-covariance $R_{\eta,mp} = \mathrm{E}\left\{\eta_m \eta_p^*\right\}$ given by

$$R_{\eta,mp} = \sigma_\eta^2 \int_{\Delta V_\mathrm{r}} d\mathbf{r}_{\mathrm{r}1} \int_{\Delta V_\mathrm{r}} d\mathbf{r}_{\mathrm{r}2} \boldsymbol{\rho}_m^\dagger(\mathbf{r}_{\mathrm{r}1}) \left[\oint d\Omega_\mathrm{r}\ \boldsymbol{\psi}^*(\mathbf{r}_{\mathrm{r}1}, \Omega_\mathrm{r}) \boldsymbol{\psi}(\mathbf{r}_{\mathrm{r}2}, \Omega_\mathrm{r})\right] \boldsymbol{\rho}_p(\mathbf{r}_{\mathrm{r}2}). \qquad (10)$$

We therefore choose $\boldsymbol{\rho}_m(\mathbf{r}_\mathrm{r})$ as the mth eigenfunction of this operation so that $R_{\eta,mp} = 0$ for $m \neq p$.

With these transmit and receive basis functions, we obtain

$$y_m = \sum_n H_{mn} x_n + \eta_m = \sqrt{R_{\eta,mm}} \left[\sum_n \widehat{H}_{mn} \hat{x}_n + \hat{\eta}_m\right] = \sqrt{R_{\eta,mm}}\, \hat{y}_m \qquad (11)$$

$$H_{mn} = \int_{\Delta V_\mathrm{r}} d\mathbf{r}_\mathrm{r} \int_{\Delta V_\mathrm{t}} d\mathbf{r}_\mathrm{t}' \boldsymbol{\rho}_m^\dagger(\mathbf{r}_\mathrm{r}) \mathbf{G}_\mathrm{rt}(\mathbf{r}_\mathrm{r}, \mathbf{r}_\mathrm{t}') \boldsymbol{\tau}_n(\mathbf{r}_\mathrm{t}'), \qquad (12)$$

If $\widehat{H}_{mn} \to 0$ as $m, n \to \infty$, then the water-filling solution will select a finite set of eigenfunctions over which to communicate and the capacity will remain bounded. However, due to supergain effects [3], these coefficients can have constant magnitude, and therefore the capacity remains unbounded. In this case, we can define a practical capacity bound by limiting the allowable supergain ratio [3], which for the transmit array is given as $Q_{\mathrm{t}n} = |x_n|^2/(|x_n|^2 \alpha A_n) = 1/(\alpha A_n)$ where α is a scale factor. For the receive array, we have $Q_{\mathrm{r}m} = 1/(\beta R_{\eta,mm})$. To avoid difficulties associated with determining α and β, we will normalize these ratios as $\widehat{Q}_{\mathrm{t}n} = A_\mathrm{max}/A_n$ and $\widehat{Q}_{\mathrm{r}m} = R_{\eta,\mathrm{max}}/R_{\eta,mm}$. We now use only the eigenfunctions whose normalized supergain ratio is below a certain threshold, denoted here as Q_0. The system capacity is then

$$C = \max_{\widehat{\mathbf{R}}_x} \log_2 \det \left[\widehat{\mathbf{H}} \widehat{\mathbf{R}}_x \widehat{\mathbf{H}}^\dagger + \mathbf{I}\right] \qquad (13)$$

where $\widehat{\mathbf{R}}_x = \mathrm{E}\left\{\hat{\mathbf{x}}\hat{\mathbf{x}}^\dagger\right\}$. The optimal covariance $\widehat{\mathbf{R}}_x$ can be determined using the water-filling solution.

This problem can be solved in closed form for currents flowing in the axial (z) direction on the surface of a circular cylinder of radius a_t and received field sampled on the surface of an identically-oriented cylinder of radius a_r. Fig. 2 plots the capacity as well as the number of eigenfunctions used as a function of Q_0 for cylinders with $a_\mathrm{t} = a_\mathrm{r} = \lambda/2$ for a line-of-sight (plane wave) propagation model. These curves emphasize the extremely large supergain ratios possible as well as the unbounded capacity growth with the number of excitation and receive modes.

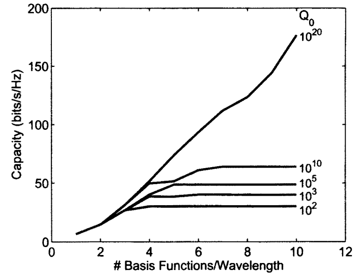

Figure 3: Numerically computed capacity versus N for the modified channel model for different values of the threshold Q_0 ($\Delta x = \Delta y = 1\lambda$).

For more general problems, we can solve the problem numerically in a fashion similar to that outlined in Section 2. Fig. 3 plots the resulting capacity versus N for an aperture with $\Delta x = \Delta y = 1\lambda$ for different values of Q_0. These results illustrate that as long as a limit is placed on the allowable supergain ratio, the capacity of the channel approaches an upper bound, with a larger allowable supergain ratio producing a higher capacity limit. Also, the number of sub-basis functions required to effectively approximate the true eigenfunctions changes depending on the threshold value Q_0. This occurs because the modes with higher supergain ratios are associated with high-frequency variations in the transmit currents (and receive weights), and therefore their representation requires a higher density of sub-basis functions.

4 Conclusion

This paper has presented a framework for determining the available capacity of continuous-space electromagnetic channels independent of the physical antennas used for different constraints on the transmit excitation and different assumptions about the receiver noise. The formulation illustrates that while antenna supergain can lead to infinite capacity, placement of practical constraints on the system leads to a finite capacity bound for fixed antenna dimensions.

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