# Visualizing clutters

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**Abstract**

If a picture is worth a thousand words, then surely clutters are among the least intutively compelling objects studied in combinatorics. What picture is conveyed by *a family of subsets of a finite set, none of which contains any other*? A large Venn diagram perhaps, in which certain unions of regions cannot be empty? In this expository note we discuss a simple technique for visualizing arbitrary clutters using graphs.

## 1. The clutter of a $K$-terminal network

A $K$*-terminal network* is a graph $G$ some of whose vertices have been designated as *terminals*; it is traditional to use $K$ to denote the set of terminals. We think of a $K$-terminal network as representing a real structure like a telephone or computer network: the terminal vertices represent the people who use the network, and the edges and non-terminal vertices represent the internal structure (cables, switching locations, etc.) of the network. Because they are useful as models of such real-world structures, $K$-terminal networks have been thoroughly studied, especially in connection with their reliability; see [1] for a discussion.

With these thoughts in mind, if we are given a $K$-terminal network $(G, K)$ we define $C(G, K)$ to be the collection of minimal sets of non-terminal vertices which are sufficient (along with their incident edges) to provide paths connecting all the terminal vertices. That is, an element of $C(G, K)$ is a minimal subset $W \subseteq V(G)-K$ such that the induced subgraph of $G$ with vertex-set $K \cup W$ is connected. For instance, if $(G, K)$ is any of the $K$-terminals networks pictured in Figure 1 then $C(G, K) = \{\{a, c\}, \{a, d\}, \{a, e\}, \{b\}, \{c, d\}, \{c, e\}\}$. (In the figure terminals are represented by large nodes and non-terminal vertices by small, lettered nodes.)

It is important to emphasize that $C(G, K)$ is defined using sets of vertices of $G$, not sets of edges. Using edges instead of vertices would result in a strictly smaller class of structures, associated with matroids; see [3, 4, 7]. If desired, an edge may be represented in $C(G,K)$ by inserting a non-terminal vertex of degree two.

In reliability theory the elements of $C(G,K)$ are referred to as *minimal pathsets of the $K$-terminal connectedness problem on* $(G,K)$; we will call them the *minpaths of* $(G,K)$ for short. Otherwise we use standard graph-theoretic terminology and notation; in particular we use $N(v)$ to denote the set of neighbors of a vertex $v$.

The fact that each element of $C(G,K)$ is a *minimal* set of non-terminal vertices sufficient to connect all the terminal vertices certainly guarantees that $C(G,K)$ must be a clutter. The purpose of this note is to discuss the result of [8] that every clutter is $C(G,K)$ for some $K$-terminal networks $(G,K)$. An important tool for proving this result, and possibly the most important topic in the elementary theory of clutters, is the *dual* or *blocker* $C^*$ of a clutter $C$: if $\emptyset \neq C \neq \{\emptyset\}$ then $C^*$ is the collection of minimal sets which intersect all the elements of $C$. The definition is completed by defining $\emptyset^* = \{\emptyset\}$ and $\{\emptyset\}^* = \emptyset$. If $(G,K)$ is a $K$-terminal network then an element of $C(G,K)^*$ is a minimal set of non-terminal vertices whose removal (along with all incident edges) will leave a subgraph of $G$ with terminals in different connected components; that is, an element of $C(G,K)^*$ is a minimal vertex cut of $G$ which consists of non-terminal vertices and whose removal will separate at least one pair of terminal vertices.

**Theorem 1** [2]**.** *If $C$ is a clutter then $C^{**} = C$.*

**Proof.** $C = \emptyset$ and $C = \{\emptyset\}$ satisfy the proposition by definition.

Suppose $C$ is a clutter of subsets of the finite set $S$ and $\emptyset \neq C \neq \{\emptyset\}$. The definition of $C^*$ implies that every element of $C$ must intersect every element of $C^*$, so every element of $C$ must contain an element of $C^{**}$.

Conversely, if $W \in C^{**}$ then $W$ intersects every $B \in C^*$ and hence $S - W$ does not contain any $B \in C^*$. By the definition of $C^*$, this implies that there is at least one $V \in C$ which does not intersect $S - W$. That is, every element of $C^{**}$ contains an element of $C$. ∎

Theorem 1 is the only result of the elementary theory of clutters that we will need here; we refer the reader to [2, 5, 6] for more thorough discussions.

**Theorem 2 [8].** *If $C$ is a clutter then there is a $K$-terminal network $(G,K)$ with $C = C(G,K)$.*

**Proof.** $C = \{\emptyset\}$ and $C = \emptyset$ correspond to edgeless networks with one or two terminals.

Suppose $|C^*| = 1$ and $\emptyset \notin C^*$; then every $W \in C$ has $|W| = 1$. If $G$ is a $K$-terminal network with $|K| = 2$ which has a non-terminal vertex for each element of an element of $C$, such that every non-terminal vertex is adjacent to all the other vertices of $G$ and the two terminal vertices are not adjacent, then $C = C(G, K)$.

If $C$ is a clutter on a set $S$ and $|C^*| \geq 2$ then we construct a $K$-terminal network $(G, K)$ with $K = C^*$ and $V(G) - K = S$ as follows: all of the non-terminal vertices of $G$ are adjacent to each other, none of the terminal vertices are adjacent to each other, and a terminal vertex $B \in C^*$ has neighbor-set $N(B) = B$, i.e., the non-terminal vertices adjacent to $B$ are the elements of $B$.

The elements of $C(G, K)$ are the minimal subsets $T \subseteq S$ such that $T \cup K$ induces a connected subgraph of $G$. Certainly such a $T$ must include at least one neighbor of each terminal vertex; by the definition of $C(G, K)$, this implies that $T \cap B \neq \emptyset$ for every $B \in C^*$. Conversely, if $T \subseteq S$ has the property that $T \cap B \neq \emptyset$ for every $B \in C^*$ then $T \cup K$ induces a connected subgraph of $G$, for the vertices in $T$ are all adjacent to each other and each terminal vertex is adjacent to at least one of them. It follows that $C(G, K)$ consists of the minimal subsets $T \subseteq S$ such that $T$ intersects every $B \in C^*$, i.e., $C(G, K) = C^{**}$. By Theorem 1, this implies that $C(G, K) = C$. ■

The proof of Theorem 2 shows that every clutter $C$ is $C(G, K)$ for a $K$-terminal network in which all non-terminal vertices are adjacent; we say such a network has a *non-terminal clique*.

## 2. $K$-terminal networks with the same clutter

As indicated in Figure 1, it is not unusual for nonisomorphic $K$-terminal networks to have the same associated clutter $C(G, K)$. Here are four simple ways to modify any $K$-terminal network without changing $C(G, K)$.

(1) If $G$ has two terminal vertices with precisely the same neighbors then one of these terminals may be removed from the network without affecting $C(G, K)$; conversely a new terminal may be introduced with the same neighbors as an existing terminal.

(2) If two terminal vertices in $G$ are adjacent, the edge connecting them may be contracted; that is, the two terminals may be combined into a single terminal which is adjacent to all the other vertices adjacent to either of the original two. Conversely, a terminal vertex may be split into two adjacent terminal vertices whose other neighbors are precisely the neighbors of the original terminal vertex.

(3) If $v$ and $w$ are terminal vertices such that neither is adjacent to any terminal, $N(v) \subseteq N(w)$, and every neighbor of $w$ not adjacent to $v$ is adjacent to every neighbor of $v$, then removing $w$ from the network will not affect $C(G, K)$; conversely one may choose any terminal $v$ not adjacent to any other terminal and introduce such a $w$.

(4) If $v$ is a non-terminal vertex which does not appear in any minpath of $(G, K)$ then any edge incident on $v$ may be removed from $G$; conversely an edge incident on $v$ may be added, so long as it does not create a minpath involving $v$.

The reader may inspect Figure 1 to see some instances of these modifications, and also to see that they are not sufficient to generate all the different $K$-terminal networks with a given $C(G, K)$.

Given a clutter $C$, it is natural to seek the simplest $K$-terminal network with $C = C(G, K)$. A reasonable definition of "simplest" involves having the smallest possible number of terminal vertices. The proof of Theorem 2 shows that if $|C^*| \geq 2$ then $C$ may be realized by a network with $|C^*|$ terminals, but it is often true that smaller networks will suffice. For instance $C = \{\{a, b, d\}, \{a, b, e, f\}, \{a, c, d\}, \{a, c, e, f\}\}$ has $C^* = \{\{a\}, \{b, c\}, \{d, e\}, \{d, f\}\}$ but only two terminals are required in a network realizing $C$. We leave it as an exercise for the reader to construct such a network. (Hint: Start with two terminals separated by a cutpoint $a$, and then separate one terminal from $a$ with a vertex cut $\{b, c\}$; separate $a$ from the other terminal with a vertex cut $\{d, e\}$, and then find a place to insert $f$ so that $\{d, f\}$ is also a vertex cut.)

Sometimes, though, $|C^*|$ terminal vertices are required. For instance, let $S = \{1, ..., n\}$ and let $C = C_{2,n}$ contain all the 2-element subsets of $S$; then $C^* = C_{n-1,n}$ consists of the $(n-1)$-element subsets of $S$. We claim that any $K$-terminal network $G$ with $C(G, K) = C$ has at least $|C^*| = n$ terminals. Suppose instead that such a $G$ has fewer than $n$ terminals; using modification (2) above we may presume that no two terminals are adjacent in $G$. There must be at least one element of $C^*$ which is not the neighbor-set of a terminal; as mentioned above, this element of $C^*$ must be a minimal vertex cut of $G$ which separates at least two terminals. Suppose $\{1, ..., n-1\}$ is such a vertex cut.

Consider the subgraph $H$ of $G$ obtained by removing the non-terminal vertices $1, ..., n-1$ and all edges incident on them. The remaining vertices include all the terminals of $G$ and in addition the non-terminal vertex $n$. If any of the terminal vertices is isolated in $H$ then all the neighbors of that terminal in $G$ are included among $1, ..., n-1$. This is impossible, because

$\{1, ..., n-1\}$ is not the neighbor-set of any terminal and if a proper subset $N \subset \{1, ..., n-1\}$ is the neighbor-set of a terminal then some subset of $N$ is a minimal vertex cut separating that terminal from another terminal; this subset of $N$ is then an element of $C^*$, contradicting the fact that all the elements of $C^*$ have cardinality $n-1$. Hence none of the terminal vertices is isolated in $H$, so all the terminal vertices are adjacent to the non-terminal vertex $n$. It follows that $H$ is connected, contradicting the assumption that $\{1, ..., n-1\}$ is a vertex cut of $G$.

## 3. Structural features of clutters

A useful visualization technique should make it easy to see structural features of the object being represented. It is a drawback of the technique we are discussing that a single clutter may be represented by many non-isomorphic $K$-terminal networks, because this guarantees that some of the information conveyed by each of the networks is irrelevant and potentially confusing. For example, in Figure 1 none of the edges between non-terminal vertices is of any significance.

An important structural feature of some clutters is the presence of *parallel elements*: if $C$ is a clutter on a set $S$ then $s, s' \in S$ are parallel with respect to $C$ if every $B \in C^*$ which contains one of $s, s'$ also contains the other. In an ordinary graph, parallel non-loop edges are not difficult to recognize: they have the same end-vertices. However, parallel non-terminal vertices in a $K$-terminal network may not be immediately noticeable. For instance, it may not be evident in Figure 1 (*iii*) - (*vi*) that $d$ and $e$ are parallel, though the similarity of their placement in Figure 1 (*i*) and (*ii*) does catch the eye. On the other hand, parallels are readily identified in networks of the type constructed in the proof of Theorem 2, for which $C(G, K)^*$ is visible as the collection of neighbor-sets of terminal vertices.

**Proposition 3.1**. *Suppose that a 1-1 correspondence between $K$ and $C(G, K)^*$ is defined by associating each $k \in K$ with $N(k)$. Then two non-terminal vertices of $(G, K)$ are parallel with respect to $C(G, K)$ if and only if they have precisely the same terminal neighbors.*

Two elements $s, s' \in S$ are *in series* with respect to a clutter $C$ if every $W \in C$ which contains one of them also contains the other. Equivalently, every $B \in C^*$ with $s \in B$ has $(B - \{s\}) \cup \{s'\} \in C^*$ and every $B' \in C^*$ with $s' \in B'$ has $(B' - \{s'\}) \cup \{s\} \in C^*$. Once again, this condition is fairly easy to recognize in $K$-terminal networks of the type constructed in the proof of Theorem 2.

**Proposition 3.2**. *Suppose that a 1-1 correspondence between $K$ and $C(G,K)^*$ is defined by associating each $k \in K$ with $N(k)$. Then two non-terminal vertices $s$ and $s'$ of $(G,K)$ are in series with respect to $C(G,K)$ if and only if whenever $s$ (resp. $s'$) is adjacent to a terminal vertex $k$, there is another terminal vertex $k'$ with neighbor-set $N(k') = (N(k) - \{s\}) \cup \{s'\}$ (resp. $N(k') = (N(k) - \{s'\}) \cup \{s\}$).*

Two operations that are important in network reliability [1] and in the combinatorial theory of clutters [5, 6] are *deletion* and *contraction*. Both of these operations are easily performed on $K$-terminal networks representing a clutter $C$. Deleting a non-terminal vertex $v$ is accomplished by removing the vertex and all its incident edges, just as one would expect. To motivate the contraction of a vertex, consider first what it means to contract an edge in an ordinary graph: the edge's function is to provide communication between its end-vertices, and after it is contracted this communication is guaranteed. In a $K$-terminal network the function of a non-terminal vertex $v$ is to provide communication among its neighbors; we contract $v$ by removing $v$ from the network and then guaranteeing communication among its neighbors by inserting an edge connecting each pair of elements of $N(v)$. If $v$ has a terminal neighbor then the effect is the same if we leave the network unchanged except for declaring that $v$ has become a terminal.

## 4. Generalizations

If the definition of $C(G,K)$ is generalized to allow non-unique labeling of the non-terminal vertices of $G$ then any clutter $C$ may be realized by a 2-terminal network. If $C^* = \{B_1, ..., B_n\}$ we may construct such a network from the disjoint union of $n$ cliques, the $i$th of which has vertices labeled by the elements of $B_i$, by making all the vertices in the $i$th clique adjacent to all the vertices in the $(i+1)$st clique for each $i < n$, and also making all the vertices in the first (resp. $n$th) clique adjacent to the first (resp. second) terminal.

Two other generalizations which may be useful are: first, to allow directed edges; and second, to consider, for $j \in \{1, ..., |K|\}$, the clutter $C_j(G,K)$ of minimal subsets $W \subseteq V(G) - K$ such that the induced subgraph of $G$ with vertex-set $K \cup W$ has no more than $j$ connected components which intersect $K$.

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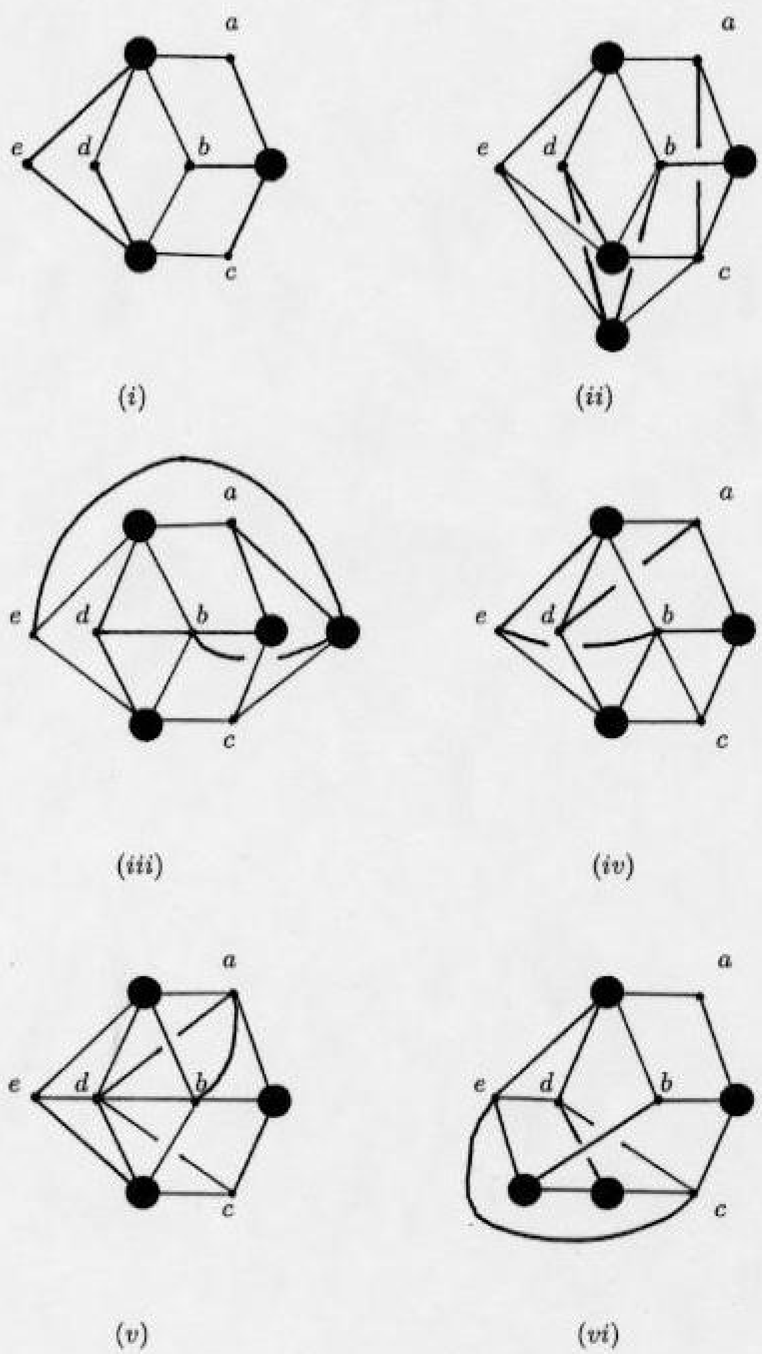


Figure 1