A note on delta-wye-delta reductions of plane graphs

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Abstract

A connected plane graph G with two distinguished vertices can be reduced to a single edge between these two vertices using certain local transformations, including series/parallel reductions and wye-delta and delta-wye transformations. Moreover a reduction consisting of these transformations can be found algorithmically in polynomial time. It follows that if an invariant of plane graphs is known to be $\#P$-hard to compute then the invariant must be fundamentally incompatible with these local transformations in some way. We discuss two examples: polynomial invariants of knots and links and the reliability of plane networks.

1. Introduction

Let G be a connected plane graph with two distinguished vertices, the *terminals* of G. Akers [1] and Lehman [13] conjectured that G may be reduced to a single edge between its terminals by a finite sequence of local transformations of the following six types: removal of loops and non-terminal vertices of degree 1, series and parallel reductions, and wye-delta and delta-wye transformations. (A wye-delta transformation replaces a non-terminal vertex of degree 3 with a circuit involving the three neighbors, and a delta-wye transformation is the inverse; see Figure 1.) Such a sequence of local transformations is a *delta-wye-delta reduction* of G. This conjecture was subsequently verified by Epifanov [5] and Grunbaum [9]. Since then Truemper [21] and Feo and Provan [6] have shown that a delta-wye-delta reduction may be found algorithmically in time $O(|V(G)|^2)$.

Colbourn, Provan and Vertigan [4] discussed the use of delta-wye-delta reductions to analyze enumeration problems on planar graphs which are traditionally solved in polynomial time using determinants, e.g., counting spanning trees. In contrast, a $\#P$-hard computation cannot be performed using delta-wye-delta reductions, which require only polynomial time; a discussion of the failure of delta-wye-delta reductions to perform the computation may lend some insight into its intractability. In this note we discuss two types of invariants of plane graphs whose computation is known to be $\#P$-hard, namely, polynomial invariants of knots and links [10] and the reliability of plane networks [16, 17]. We refer the interested reader to [3, 12, 22] for detailed introductions to these invariants.

2. Polynomial invariants of knots and links

A *knot* is a smooth, simple closed curve in 3-space, and a *link* is the union of a finite number of pairwise disjoint knots. A knot or link may be placed in standard position, so that projection on the plane yields only a finite number of singularities, all of which are double points or *crossings*; a projection is made into a *diagram* by indicating which of the two arcs incident on each crossing is the overpassing arc. Goeritz [8, 18] associated a plane graph G to a knot or link diagram by 2-coloring the diagram's complementary regions: the unbounded region is colored light, and each other region is colored light or dark in such a way that every arc of the diagram separates differently-colored regions. G is to have a vertex inside each dark-colored region and an edge for each crossing of the diagram, connecting the vertex or vertices inside the dark-colored region(s) incident on that crossing. An edge is assigned a weight $+1$ or -1 according to whether the overpassing arc of the corresponding crossing is on the right or left side of the dark region(s) incident on that crossing, as seen by a person standing on the crossing. Some examples of link diagrams and their associated Goeritz graphs are given in Figure 2.

Consider the knot-theoretic significance of a delta-wye-delta reduction of the Goeritz graph G of a knot or link diagram. It may happen that a local transformation of G corresponds to a trivial change in the diagram, i.e., a change that does not affect the knot or link type represented. For instance, removing the loop in Figure 2 (c) corresponds to the elimination of an unnecessary crossing in a diagram of a Hopf link. However, it may also happen that a local transformation is not trivial; for example, replacing the two parallel edges of Figure 2 (b) by a single edge as in Figure 2 (a) has the effect of changing a figure-eight knot to a trefoil knot. Such operations are part of the calculus of polynomial invariants of knots and links. For instance, the Kauffman polynomials [11] of the four links pictured in Figure 2 satisfy the equation $L_{(b)} = z \cdot (L_{(a)} + L_{(c)}) - L_{(d)}$. As the knot pictured in part (d) can easily be simplified (it is unknotted) and those in parts (a) and (c) have only three crossings, this equation is part of the process of calculating $L_{(b)}$ recursively, using the Kauffman polynomials of simpler knots and links. On the other hand, the reduction of Figure 2 (b) to Figure

2 (a) is incompatible with the homfly polynomial [7], as it is inconsistent with the string orientations of the knots.

We see two reasons that knot-theoretic calculations are $\#P$-hard. Some knot-theoretic calculations are simply incompatible with some of the local transformations that appear in delta-wye-delta reductions. Other knot-theoretic calculations are compatible with all of the local transformations, but require that certain local transformations be modified to produce several terms rather than just one; the number of terms that ultimately result is exponential in the number of these modified local transformations.

3. Network reliability

Suppose now that we attempt to use a delta-wye-delta reduction to calculate the reliability of a two-terminal plane network G. We presume that each edge e of G is given with a certain probability $p(e)$ of successful operation and that the edges operate independently of each other; the reliability $Rel(G)$ is the probability of successful operation of at least one path connecting the two terminal vertices of G. It is a simple matter to verify that the first four types of local transformations involved in delta-wye-delta reductions (removal of loops, removal of non-terminal vertices of degree 1, and series and parallel reductions) can all be accomplished so that the transformed network has precisely the same reliability as the original one. For instance, if parallel edges e_1 and e_2 are to be replaced with a single edge e, then e should have the same probability of successful operation as the event "e_1 operates or e_2 operates," namely $p(e_1) + p(e_2) - p(e_1)p(e_2)$.

However the delta-wye and wye-delta transformations are more complicated. If the reliability of the entire network is to be unchanged by a delta-wye or wye-delta transformation, the delta and wye should provide the same probability that any two of v_1, v_2, v_3 may communicate within Figure 1, or that all three vertices may communicate. If we label the edges of the delta e_{12}, e_{13}, e_{23} so that e_{ij} is incident on v_i and v_j, and we label the edge of the wye incident on v_i as e_i^*, then we are led to the following system of equations.

$$\begin{gathered}
p(e_{12}) + (1 - p(e_{12}))p(e_{13})p(e_{23}) = p(e_1^*)p(e_2^*) \\
p(e_{13}) + (1 - p(e_{13}))p(e_{12})p(e_{23}) = p(e_1^*)p(e_3^*) \\
p(e_{23}) + (1 - p(e_{23}))p(e_{12})p(e_{13}) = p(e_2^*)p(e_3^*) \\
p(e_{12})p(e_{13}) + p(e_{12})(1 - p(e_{13}))p(e_{23}) + (1 - p(e_{12}))p(e_{13})p(e_{23}) \\
= p(e_1^*)p(e_2^*)p(e_3^*)
\end{gathered}$$

Note that given a delta or a wye, finding the edge-probabilities of an equivalent wye or delta requires the solution of a system of four equations in three unknowns; it will come as no surprise that this is impossible for almost all combinations of parameter values [13, 20]. (Solving *some* of the equations can be useful in transforming a network containing a given wye or delta into networks whose reliabilities bound that of the original network [13]; in many examples these bounds result in very good approximations [2].)

It has been suggested that the definition of *plane network* be modified so that the vertices [19] and faces [20] of the network have probabilities of successful operation, just as the edges do. (The failure of a vertex or face makes the incident edges useless.) When considering a delta-wye or wye-delta transformation of such a network it is important to realize that the operational probabilities of vertices and faces that appear in both the delta and the wye are not variables: they are given when a delta or wye is specified, and must stay the same in an equivalent wye or delta. These modified plane networks have one extra variable on each "side" of a delta-wye or wye-delta transformation, corresponding to the extra vertex in the wye and the extra face in the delta; these extra variables will generally allow for the solution of the corresponding system of equations. As discussed in [20], this new system of equations will have the following form.

$$\begin{aligned} p\cdot(s_{12}+s_3(1-s_{12})s_{13}s_{23}) &= rs_1^*s_2^* \\ p\cdot(s_{13}+s_2(1-s_{13})s_{12}s_{23}) &= rs_1^*s_3^* \\ p\cdot(s_{23}+s_1(1-s_{23})s_{12}s_{13}) &= rs_2^*s_3^* \\ p\cdot(s_{12}s_{13}+s_{12}(1-s_{13})s_{23}+(1-s_{12})s_{13}s_{23}) &= rs_1^*s_2^*s_3^* \end{aligned} \qquad (3.1)$$

If we label the faces of G incident on the wye as f_{12}, f_{13}, f_{23} in the obvious way then the significance of the variables in (3.1) depends on whether or not any of f_{12}, f_{13}, f_{23} happen to coincide. For instance, if $f_{12} \neq f_{13} \neq f_{23} \neq f_{12}$ then $r = p(v_0)p(f_{12})p(f_{13})p(f_{23})$, $p = p(f_0)$ is the operational probability of the central face f_0 of the delta, each s_i^* is $p(e_i^*)$, each s_i is $p(v_i)$, and each s_{ij} is $p(e_{ij})p(f_{ij})$. On the other hand, if $f_{12} = f_{13} = f_{23}$ then $r = p(v_0)$ and each s_{ij} is $p(e_{ij})$. We refer to [20] for a detailed discussion.

Given a delta, it is not difficult to see how these equations may be solved to give the probabilities of successful operation of the components of an equivalent wye. To find s_1^*, for instance, we divide the left-hand side of the fourth equation of (3.1) by the left-hand side of the third. This will often result in "probabilities" which are >1 or <0, but these can usually be dealt with formally without difficulty. In some examples, though, the left-hand side of one of the equations may be 0, and this may make it impossible to solve (3.1).

If we are given a wye then the process of solving (3.1) to find an equivalent delta is more complicated. Almost all combinations of parameter values do not allow solutions with $1/s_1 = s_{12}$ or $1/s_2 = s_{12}$; for this reason we called such solutions *sporadic* in [20]. To find the solutions with $1/s_1 \neq s_{12} \neq 1/s_2$ we multiply the last equation of (3.1) by $-s_2 s_3$, the first by s_2, and the second by s_3; adding these three yields the formula

$$s_{13} = \frac{r s_1^*(s_2 s_2^* + s_3 s_3^* - s_2 s_2^* s_3 s_3^*) - p s_2 s_{12}}{p s_3 (1 - s_2 s_{12})}.$$

A similar formula may be derived for s_{23}, and using these formulas we reduce the four equations of (3.1) to two equations in the unknowns p and s_{12}. One of these two equations is quadratic in p. We may use the quadratic formula to solve this one symbolically for p in terms of s_{12}; the other equation then results in two equations in which s_{12} is the only unknown, and solving either of these latter two equations gives a solution of (3.1). These two resulting equations include the 6^{th} power of s_{12} and the square root of a degree-4 polynomial in s_{12}. The square root may be removed by collecting on one side of each equation those terms that involve the square root, and then squaring. The result is a pair of degree-12 polynomial equations in s_{12}, and these cannot be solved exactly in general, though accurate approximate solutions may be found.

This solution process is discontinuous where a denominator of 0 appears in the quadratic formula or in the formula for s_{13} or s_{23}. For instance if we consider a wye with $s_1^* = .05$, $s_2^* = .04$, $s_3^* = .03$, $s_1 = .03$, $s_2 = .04$, $s_3 = .05$ and $r = .05$ then according to *Mathematica* there are five equivalent deltas: three occur when $+$ is used in the quadratic formula (with s_{12} approximately 0.013, 2.020, and 32.454), and two occur when $-$ is used in the quadratic formula (with s_{12} approximately 0.517 and 25.614). If we change the wye by changing s_3^* and s_1 to -0.03 then there are again five solutions, three occurring when $+$ is used in the quadratic formula (with s_{12} approximately 0.033, -1.111, and 760) and two occurring when $-$ is used in the quadratic formula (with s_{12} approximately 0.493 and 990.585). Both of the solutions with $s_{12} \sim 0$ yield values for s_{13}, s_{23} and p that are close to 0, but none of the other solutions for one wye is close to any of the solutions for the other. (The two solutions with $s_{12} \sim 0.5$ yield s_{13}-values close to 26 and 750, respectively.) There are similar examples for which none of the original wye's parameters are particularly close to 0, so this kind of behavior is not limited to the immediate neighborhood of the origin.

In sum, we see two reasons for the intractability of calculating the reliability of a plane network. One reason is that it will occasionally happen that (3.1) is simply unsolvable, e.g., if one side of one of the equations is 0. The second reason is that solving (3.1) to find a delta equivalent to a given wye is neither an exact nor a continuous function of the parameters of the wye, and hence small errors introduced when approximate solutions are used in one stage of a delta-wye-delta reduction may occasionally result in large errors in later stages of the reduction, unless computationally expensive techniques are used to bound such accumulation of errors. The fact that the wye-delta transformation is particularly troublesome is also evident in results like those of [14, 15], in which a polynomial-time reliability algorithm is given for networks which may be reduced using modified delta-wye transformations and the simpler local transformations (loop removal, series/parallel reduction, and removal of non-terminal vertices of degree 1).

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Figure 1

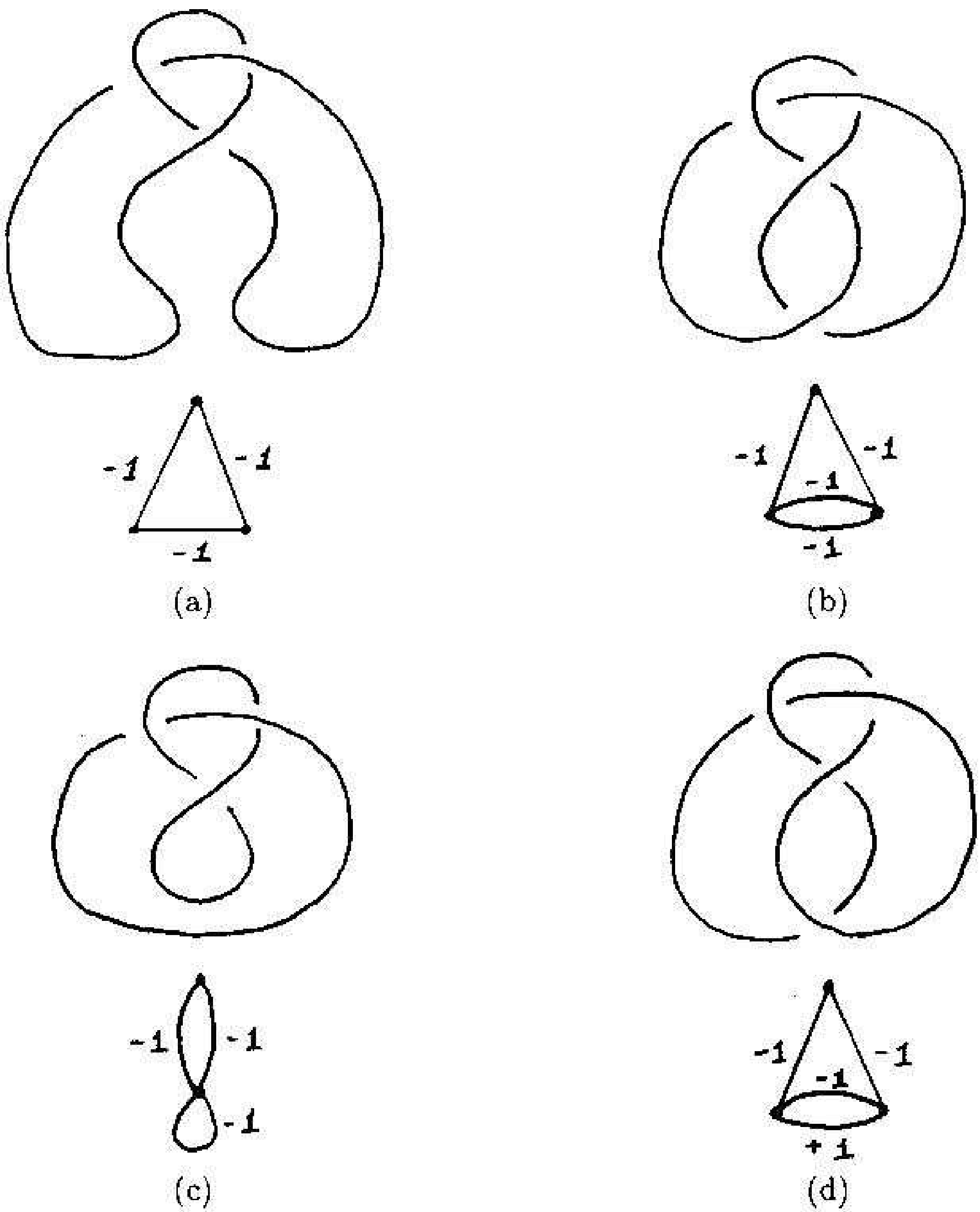

Figure 2