Two universal reliability problems

Lorenzo Traldi
Department of Mathematics, Lafayette College
Easton, Pennsylvania 18042

Abstract

A type of reliability problem is defined to be *universal* if every reliability problem is equivalent to one of that type. We give examples of two universal reliability problems, distributed program reliability and K-terminal connectedness.

Let E be a finite set. By a *reliability problem* on E we mean a function $p : E \to \mathbb{R}$ together with a clutter C of subsets of E; that is, C is a family of subsets of E and no element of C contains any other element of C. The elements of C are the *minpaths* of the reliability problem, and a subset of E which contains a minpath is a *pathset* or *operational state.* (Implicit in this definition is the assumption that the reliability problems we consider are *coherent*, i.e., every subset of E which contains a minpath is considered an operational state.) The *reliability* of a reliability problem is the probability that a randomly chosen subset of E will contain a subset of C,

$$Rel(C) = \sum_{S \in \mathfrak{F}(C)} (\prod_{e \in S} p(e))(\prod_{e \notin S}(1 - p(e))),$$

where $\mathfrak{F}(C) = \{S \subseteq E \,|\, S \text{ contains an element of } C\}$ is the *filter* associated to C.

Using language somewhat informally we will also sometimes refer to a type of reliability problem as "a reliability problem". The most familiar example of a reliability problem is the all-terminal network reliability problem: given a connected graph G we let $E = E(G)$ and $C = \{$ edge-sets of spanning trees of $G\}$, so that $\mathfrak{F}(C)$ consists of the edge-sets of connected spanning subgraphs of G.

In this paper we introduce the idea of a *universal reliability problem.* By this we mean a type of reliability problem with the property that every reliability problem on every finite set is isomorphic to some problem of that type.

There are several reasons we hope this idea will prove to be of interest. One is that in providing concrete situations in which all finite reliability

CONGRESSUS NUMERANTIUM **132** (1998), pp.199-204

problems can arise, universal problems may make it easier to arrive at certain insights or conclusions which are valid for finite reliability problems generally. Another is that when a particular reliability problem is identified as not being universal, one may be able to focus attention on special characteristics of that problem which will be useful in analyzing it. Finally, when a particular problem is identified as being universal it follows that techniques known to apply to only a limited variety of reliability problems cannot be successfully applied to it.

The ideas in this paper were inspired by a question of A. Satyanarayana, who once asked the author whether a version of reliability domination (see [1, 2, 3, 5, 10, 11, 13] for accounts of this important theory) could be formulated to apply to the distributed program reliability problem studied in [9]. It follows from our first and third theorems that the answer to Satyanarayana's question is "no".

Theorem 1. *The distributed program reliability problem is universal.*

Another type of reliability problem mentioned in the literature is the K-terminal connectedness problem [4].

Theorem 2. *The K-terminal connectedness problem is universal.*

The reader familiar with the theory of network reliability domination introduced by Satyanarayana and his coauthors [10, 11, 13] may recall that it has been generalized to apply to *totally amenable* reliability problems [2, 8], and that not all reliability problems are totally amenable. This last fact directly implies the following theorem, which we will discuss in more detail in Section 2.

Theorem 3. *The theory of reliability domination does not "work" for universal reliability problems.*

1. Two universal problems

Distributed program reliability was introduced in [9] to model a certain kind of computer processing system. We are given a graph G, with a single distinguished vertex. The distinguished vertex represents a computer processor, which requires access to certain sets of files in order to run a program. The other vertices represent sites at which files are stored. Edges of the graph represent communication links. The problem is determined by specifying the sets of files which suffice for the operation of the program, the sets of files stored at the various vertices and the probabilities of successful operation of the edges and vertices of the graph.

To prove Theorem 1 we will show that if E is any finite set and C is any clutter of subsets of E then there is a distributed program reliability problem on a graph G in which $E = E(G)$, the vertices of G are all *perfect* (i.e., they are guaranteed to operate), and C is the clutter of minimal sets of edges of G which provide the processor node access to all files.

Our proof uses one of the basic notions of the combinatorics of clutters, the *dual* or *blocker* of a clutter; see [6] for more information on this notion. If C is a clutter on E then the dual of C is $C^* = \{$ minimal subsets of E which intersect all elements of C $\}$; the elements of C^* are called the *mincuts* of C in reliability theory. It will be important for us to recall that $C^{**} = C$.

Suppose C is a clutter on a finite set E, and let $C^* = \{K_1, ..., K_n\}$. Let G be a graph with vertex-set $V(G) = \{v_0\} \cup \{v_e \mid e \in E\}$ and edge-set $E(G) = E$; the edge e is to connect v_0 to v_e. Only the edges of G are subject to failure. We set up a distributed reliability program on G which represents a computer whose processor is located at v_0, and which requires access to files $F_1, ..., F_n$. These files are located at the vertices of G other than v_0, with F_i located at v_e if and only if $e \in K_i$. A subset S of E is an operational state of the reliability problem if and only if it provides the processor at v_0 access to every file F_i; this will happen only if $K_i \cap S \neq \emptyset$ $\forall i \in \{1, ..., n\}$. Consequently the minpaths of the reliability problem are the minimal subsets of E which intersect every K_i; that is, they are the elements of C^{**}. As $C^{**} = C$, this completes the proof of Theorem 1.

A K-terminal connectedness problem is another kind of reliability problem based on a graph G [4]. To specify such a problem we specify a subset K of the vertex-set $V(G)$. Vertices not in K are subject to failure; elements of K and all edges of G are perfect. The operational states of the reliability problem are the subsets $S \subseteq V(G) - K$ such that K is contained a single component of the full subgraph of G on $S \cup K$. That is, the operational states are subsets of $V(G) - K$ sufficient to provide communication among all the elements of K.

To show that the K-terminal connectedness problem is universal we must show that if C is any clutter on a finite set E then there is a graph G with $V(G) = E \cup K$ such that C is the clutter of minpaths of the associated K-terminal connectedness problem. Let $C^* = \{K_1, ..., K_n\}$. G will have $V(G) = E \cup K$, where $K = \{k_1, ..., k_n\}$. No two elements of K are to be adjacent to each other in G, and all the elements of E are to be adjacent to each other; also, G is to have an edge connecting e to k_i if and only if $e \in K_i$. For a subset $S \subseteq E$ to be an operational state of this K-terminal connectedness problem, every k_i must be adjacent to some element of S, i.e., $K_i \cap S \neq \emptyset$ for every i. Consequently the minpaths are the elements of

$C^{**} = C$.

2. Reliability domination

Reliability domination is one of the most important theoretical advances in network reliability of the last twenty years. We will not discuss the theory in any detail here, but refer the reader to the many presentations in the literature [1, 3, 5, 11]. An integer, the *domination*, is attached to each reliability problem; it is useful in measuring the complexity of a certain kind of algorithm used to calculate the reliability of the problem. The domination of a reliability problem C on E is defined by the following formula.

$$d(C) = \sum_{S \in \mathfrak{F}(C)} (-1)^{|S|}$$

(This is equivalent to the more familiar and complicated definition in terms of "formations" that is usually cited [7].)

A fundamentally important property of the domination of an all-terminal network reliability problem is the fact that the domination is zero if and only if some edge of the network is a loop, i.e., it doesn't appear in any spanning tree. This property is also true in other contexts in which the theory of reliability domination has been successfully applied, e.g., K-terminal reliability and, most generally, totally amenable reliability.

To prove Theorem 3 we assert that in contrast, any universal problem must have examples which have $d(C) = 0$ even though every element of E appears in some element of C. To verify this assertion, note that the clutter $P = \{\{a,b\},\{b,c\},\{c,d\}\}$ on $E = \{a,b,c,d\}$ has

$$d(C) = \sum_{S \in \mathfrak{F}(C)} (-1)^{|S|} = 3 \cdot (-1)^2 + 4 \cdot (-1)^3 + (-1)^4 = 0.$$

By the way, P is one of the forbidden minors for matroid ports found by Seymour [14]. In Figure 2.1 we give an example of a distributed program reliability problem for which P is the clutter of minpaths. The computer processor is located at the vertex v_0 and the computer requires access to two files, F_1 (located at vertices v_1 and v_2) and F_2 (located at vertices v_3 and v_4).

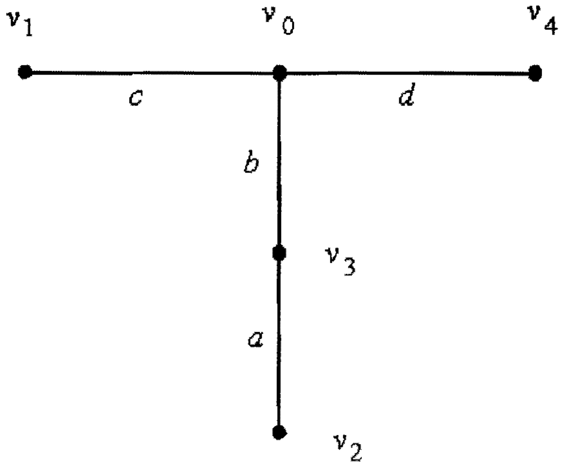

Figure 2.1

References

[1] A. Agrawal and R. E. Barlow, *A survey of network reliability and domination theory*, Oper. Res. **32** (1984), 478-492.

[2] R. E. Barlow and S. Iyer, *Computational complexity of coherent systems and the reliability polynomial*, Prob. Eng. Inf. Sci. **2** (1988), 461-469.

[3] F. T. Boesch, A. Satyanarayana and C. L. Suffel, *Some recent advances in reliability analysis using graph theory: a tutorial*, Congr. Numer. **64** (1988), 253-276.

[4] F. T. Boesch, X. Li and J. Rodriguez, *Graphs with most number of three point induced connected subgraphs*, Discr. Appl. Math. **59** (1995), 1-10.

[5] C. J. Colbourn, *The combinatorics of network reliability*, Oxford Univ. Press, Oxford, 1987.

[6] J. Edmonds and D. R. Fulkerson, *Bottleneck extrema*, J. Comb. Theory **8** (1970), 299-306.

[7] A. B. Huseby, *Domination theory and the Crapo β-invariant*, Networks **19** (1989), 135-149.

[8] A. B. Huseby, *On regularity, amenability and optimal factoring strategies for reliability computations*, preprint, University of Oslo, Oslo, Norway.

[9] V. K. Prasanna Kumar, S. Hariri, and C. S. Raghavendra, *Distributed program reliability analysis*, IEEE Trans. Software Eng. **SE-12** (1986), 42-50.

[10] A. Satyanarayana, *A unified formula for the analysis of some network reliability problems*, IEEE Trans. Reliability **R-31** (1982), 23-32.

[11] A. Satyanarayana and M. K. Chang, *Network reliability and the factoring theorem*, Networks **13** (1983), 107-120.

[12] A. Satyanarayana and Z. Khalil, *On an invariant of graphs and the reliability polynomial*, SIAM J. Alg. Disc. Meth. **7** (1986), 399-403.

[13] A. Satyanarayana and A. Prabhakar, *A new topological formula and rapid algorithm for reliability analysis of complex networks*, IEEE Trans. Reliability **R-27** (1978), 82-100.

[14] P. D. Seymour, *A forbidden minor characterization of matroid ports*, Quart. J. Math. Oxford (2) **27** (1976), 407-413.