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A NOTE ON A THEOREM OF WU

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Abstract

Wu proved that the Arf invariant of a proper link L is determined by the reduced potential function $D_L(z)$ and the Arf invariants of certain sublinks of L. We observe that $D_L(z)$ detects the existence of such sublinks.

The *Arf invariant* of a tame link $L = K_1 \cup \cdots \cup K_\mu$ in the three-sphere was introduced by Robertello [7]. It is a mod 2 integer which is defined only if L is *proper*, i.e., either $\mu = 1$ or else $\mu \geq 2$ and for every $i \in \{1, \ldots, \mu\}$ the linking number $\ell(K_i, L - K_i)$ is even. In the years since the Arf invariant was introduced, it has been shown that it can be defined recursively using the reduced potential function of L, $D_L(z) = z^{\mu-1} \cdot (a_0(L) + a_2(L)z^2 + a_4(L)z^4 + \cdots)$. If $\mu = 1$ then $\mathit{Arf}(L) \equiv a_2(L) \pmod 2$ [3, 4, 5], and if $\mu > 1$ then

$$\mathit{Arf}(L) \equiv a_2(L) + \sum \mathit{Arf}(L') \pmod 2,$$

with a summand for each doubly proper sublink L' of L [8]. (A sublink $L' \subset L$ is *doubly proper* if both L' and $L - L'$ are nonempty proper links.)

In this note we observe that $a_0(L)$ detects the existence of doubly proper sublinks.

THEOREM. *A proper link L has doubly proper sublinks if and only if $a_0(L)$ is even.*

PROOF. If $\mu = 1$ then $a_0(L) = 1$ and L has no doubly proper sublinks.

If $\mu = 2$ then $a_0(L) = \ell(K_1, K_2)$ is even and L has two doubly proper sublinks, its components.

If $\mu > 2$ and $\lambda(L)$ is the $(\mu-1) \times (\mu-1)$ matrix with $\lambda(L)_{ij} = -\ell(K_i, K_j)$ for $i \neq j$ and $\lambda(L)_{ii} = \ell(K_i, L - K_i)$ then $a_0(L) = \det \lambda(L)$ [1, 2], and consequently the mod-2 reduction $\lambda_2(L)$ of $\lambda(L)$ has $a_0(L) \equiv \det \lambda_2(L) \pmod 2$. Note that the diagonal entries of $\lambda_2(L)$ are all 0.

If $a_0(L)$ is even then $\lambda_2(L)$ is singular, so there must be $1 \leq i_1 < \cdots < i_\nu \leq \mu - 1$ such that the sum of the $i_1^{th}, \ldots, i_\nu^{th}$ columns of $\lambda_2(L)$ is 0. It follows immediately that if $L' = K_{i_1} \cup \cdots \cup K_{i_\nu}$ then $\ell(K_{i_j}, L' - K_{i_j})$ is even for $1 \leq j \leq \nu$ (i.e., L' is a proper link) and also that $\ell(K_i, L')$ is even for $i \notin \{i_1, \ldots, i_\nu\}$; the latter implies that $\ell(K_i, L - L' - K_i) = \ell(K_i, L - K_i) - \ell(K_i, L')$ is even for $i \notin \{i_1, \ldots, i_\nu\}$ (i.e., $L - L'$ is a proper link). Consequently L' is a doubly proper sublink of L.

Conversely, if L has a doubly proper sublink L' then by considering $L - L'$ instead of L' if necessary, we may presume that $L' = K_{i_1} \cup \cdots \cup K_{i_\nu}$ with $i_\nu < \mu$. If we add up the $i_1^{th}, \ldots, i_\nu^{th}$ columns of $\lambda(L)$ then each entry of the sum is even, because each entry is either $\ell(K_i, L' - K_i)$ or $\ell(K_i, L') = \ell(K_i, L - K_i) - \ell(K_i, L - L' - K_i)$, according to whether $K_i \subseteq L'$ or not. Consequently $\lambda_2(L)$ is singular, and $a_0(L)$ is even. ■

The following corollaries generalize some results of Murakami [6] and Wu [8].

COROLLARY 1. *If L is a proper link and $a_0(L)$ is odd then $Arf(L) \equiv a_2(L)$* (mod 2).

COROLLARY 2. *If L is a proper link and μ is even then L has doubly proper sublinks.*

PROOF. The mod-2 reduction $\lambda_2(L)$ of $\lambda(L)$ has zeroes on its diagonal and is a (skew) symmetric matrix of odd size, so it must be singular. ■

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