MATHEMATICS SEMINAR NOTES
Vol. 11 (1983), 363–380

SOME PROPERTIES OF THE DETERMINANTAL IDEALS OF LINK MODULES

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(Received March 7, 1983)

Let $L=K_1\cup\cdots\cup K_\mu\subseteq S^3$ be a tame link of $\mu\geq 1$ components with group $G=\pi_1(S^3-L)$. Also, let $H=G/G'$ be the abelianization of G; H is then the (multiplicative) free abelian group generated by the meridians $t_1,\dots,t_\mu$ of L, and its integral group ring ZH consists of polynomials (with integer coefficients) in $t_1,\dots,t_\mu,t_1^{-1},\dots,t_\mu^{-1}$. The homomorphism $\varepsilon\colon ZH\to Z$ (which has $\varepsilon(t_i)=1$ for each i) is the augmentation map, and its kernel is the augmentation ideal IH of ZH.

If $p\colon\tilde{X}\to X=S^3-L$ is the universal abelian cover of X, and F is its fiber, then

$$0=H_1(F;Z)\longrightarrow H_1(\tilde{X};Z)\longrightarrow H_1(\tilde{X},F;Z)$$
$$\longrightarrow H_0(F;Z)\longrightarrow H_0(\tilde{X};Z)\longrightarrow H_0(\tilde{X},F;Z)=0$$

is a portion of the long exact homology sequence of the pair $(\tilde{X},F)$. H can be canonically identified with the group of covering automorphisms of $\tilde{X}$, and this identification leads naturally to H-module structures on the homology groups. In particular, $H_0(F;Z)\cong ZH$ and $H_0(\tilde{X};Z)\cong Z$ as H-modules; furthermore, the isomorphisms can be chosen so that their composition with the map $H_0(F;Z)\to H_0(\tilde{X};Z)$ induced by inclusion is the augmentation map. Thus we obtain the short exact sequence

$$0\longrightarrow H_1(\tilde{X};Z)\longrightarrow H_1(\tilde{X},F;Z)\longrightarrow IH\longrightarrow 0$$

of ZH-modules; this is the *module sequence of L*. This sequence is discussed in greater depth in [2].

The ZH-module $H_1(\tilde{X},F;Z)$ has been called the *Alexander module of* L [2]; we will denote it A_L. The ZH-module $H_1(\tilde{X};Z)$ has been called the *Alexander invariant of* L [6]; we will denote it B_L.

Both these ZH-modules are finitely presented, and so have well defined determinantal ideals $E_k(A_L)$, $E_k(B_L)\subseteq ZH$ for each integer k. (The algebraic theory of these invariants, also known as elementary ideals or Fitting invariants, is discussed in [5].) In previous papers [9; 10] we have discussed various properties of the elementary ideals of A_L (in both the notation $E_k(A_L)=E_k(L)$ was used). Our aim here is to find analogous properties of those of B_L.

If $\mu\geq 2$, let L_μ be the sublink $L_\mu=L-K_\mu$ of L, G_μ its group, and H_μ the abelianization of G_μ; let ϕ: $ZH \to ZH_\mu$ be the homomorphism given by $\phi(t_i)=t_i$ for $i<\mu$, and $\phi(t_\mu)=1$. In [9] the relationship between the determinantal ideals of A_{L_μ} and the images, under ϕ, of those of A_L was considered; our first theorem here is concerned with the relationship between the determinantal ideals of B_L and B_{L_μ}.

THEOREM 1_2: *If $\mu=2$, then for any value of k*

$$E_k(B_{L_\mu}) \supseteq \phi E_k(B_L) \supseteq \left(\frac{t_1^{\ell_1}-1}{t_1-1}\right)E_k(B_{L_\mu})+E_{k-1}(B_{L_\mu}).$$

THEOREM 1_μ: *If $\mu \geq 3$, then for any value of k*

$$E_k(B_{L_\mu}) \supseteq \phi E_k(B_L) \supseteq$$

$$((\prod_{j=1}^{\mu-1} t_j^{\ell_j})-1)\,(IH_\mu)^{\mu-3}E_k(B_{L_\mu}) + \sum_{i=0}^{\mu-2}(IH_\mu)^i E_{k-\mu+1+i}(B_{L_\mu})$$

$$+\sum_{i=1}^{\mu-2}\sum_{j=1}^{\mu-1}(\ell_j)(\{t_r-1 \mid j\neq r\})^{i-1}E_{k-\mu+1+i}(B_{L_\mu})$$

$$+\sum_{i=2}^{\mu-2}\sum_{p=2}^{i}\Sigma((\prod_{q=1}^{p} t_{j_q}^{\ell_{j_q}})-1)\,(\{t_{j_q}-1\})^{p-2}(\{t_r-1 \mid r\neq j_q \forall q\})^{i-p}E_{k-\mu+1+i}(B_{L_\mu}),$$

where the sum Σ is taken over the set of all p-tuples $(j_1,\dots,j_p)$ with $1\leq j_1<\cdots<j_p<\mu$. (If $\mu=3$ the final sum is 0.)

(For $0<j<\mu$, we have used ℓ_j to denote the linking number $\ell(K_j, K_\mu)$. Also, for any ideal D of ZH, $D^0=ZH$.)

It follows from Torres' second relation [7] and the work of Crowell and Strauss [3] that the second inclusion of Theorem 1 is an equality for $k=0$ and any $\mu\geq 2$. However, for specific links L and values of $k\geq 1$ it is possible for either inclusion of Theorem 1 to be an equality (without the other's being one), or neither, or both. For instance, calculations which we will not present here indicate that all four possibilities are displayed by the links of [9, Examples 1 through 5].

If $K\subseteq S^3$ is a tame knot (i.e., a one-component link), then it is well known that $E_0(B_K)$, the princinpal ideal of ZH generated by the Alexander polynomial of K, has the property that $\varepsilon E_0(B_K)=Z$ [6, p. 207]. A simple inductive argument, using this fact and Theorem 1, is sufficient to verify

COROLLARY 1: $\varepsilon E_k(B_L)=Z$ whenever $k\geq\binom{\mu}{2}$.

For $p\in Z$ we have used $\binom{p}{2}$ to denote the binomial coefficient; in particular, $\binom{p}{2}=0$ for $p\leq 1$. If $\mu\geq 2$, let $\lambda=(\lambda_{i(p,q)})$ be the $\mu\times\binom{\mu}{2}$ matrix whose rows and columns are indexed by the sets $\{1,\dots,\mu\}$ and $\{(p,q)\mid 1\leq p<q\leq\mu\}$, respectively, and whose entries are given by

$$\lambda_{i(p,q)} = \begin{cases} \ell(K_i, K_q) & \text{if } i = p \\ -\ell(K_i, K_p) & \text{if } i = q \\ 0 & \text{if } p \neq i \neq q. \end{cases}$$

If $\mu=1$, on the other hand, then let λ be the one-by-one matrix whose lone entry is 1.

THEOREM 2: *If* $\mu \geq 2$ *and* $1 \leq k \leq \binom{\mu}{2}$ *then*

$$\sum_{i=0}^{k-1} E_{\binom{\mu}{2}-k+i}(B_L)\cdot(IH)^i + (IH)^k = \sum_{i=0}^{k-1} E_{\binom{\mu}{2}-k+i}(\lambda)\cdot(IH)^i + (IH)^k.$$

Since the rows of λ are linearly dependent if $\mu \geq 2$, $E_j(\lambda)=0$ whenever $j \leq \binom{\mu}{2}-\mu$. Combining this observation with Theorem 2, we obtain

COROLLARY 2: $E_{\binom{\mu-1}{2}-k}(B_L) \subseteq (IH)^k$ *for any* $k \geq 1$.

Also, combining Theorem 2 and Corollary 1 we conclude

COROLLARY 3: *For any* $k \in \mathbf{Z}$, $\varepsilon E_k(B_L) = E_k(\lambda)$.

In this respect, the behavior of the determinantal ideals of B_L is quite different from that of those of A_L, since (as noted in [9, §1]) the augmented ideals $\varepsilon E_k(A_L)$ are completely determined by μ.

Recall that the *lower central series subgroups of* G are given by $G_1 = G$ and, for $q \geq 1$, $G_{q+1} = [G_q, G]$. Our proof of Theorem 2 is based on the work of K. T. Chen, who has shown [1, Corollary 2] that the matrix λ is a presentation matrix for the abelian group G_2/G_3, for any $\mu \geq 1$. Using the well-known structure theorem for finitely generated abelian groups, it follows that the ideals $E_k(\lambda) \subseteq \mathbf{Z}$ form a complete set of invariants of G_2/G_3. Thus Corollary 3 may be stated in the following alternate form.

COROLLARY 4: *Let* H *and* $\tilde{H}$ *be the abelianizations of the groups* G *and* $\tilde{G}$ *of two tame links* L *and* $\tilde{L}$ *in* S^3; *also, let* $\varepsilon: \mathbf{Z}H \to \mathbf{Z}$ *and* $\tilde{\varepsilon}: \mathbf{Z}\tilde{H} \to \mathbf{Z}$ *be the augmentation maps. Then the abelian groups* G_2/G_3 *and* $\tilde{G}_2/\tilde{G}_3$ *are isomorphic if, and only if,* $\varepsilon E_k(B_L) = \tilde{\varepsilon} E_k(B_{\tilde{L}}) \forall k \in \mathbf{Z}$.

Note that Corollaries 1, 2, 3, and 4 hold for any $\mu \geq 1$, while Theorems 1 and 2 require $\mu \geq 2$.

It should be remarked that, although the results of this paper show that the determinantal ideals of the $\mathbf{Z}H$-modules A_L and B_L have some analogous properties, we will not actually discuss the relationship between these two sequences of determinantal ideals. The reader interested in this relationship (which is not completely understood) is referred to [3; 8]. In addition, we should remark that Theorem 2 can be generalized to a relation between the elementary ideals of B_L and the Milnor invariants of L [11].

The author would like to thank the referee who read [9], and suggested the line of inquiry that led to Theorem 1. Also, we must thank William S. Massey, whose comments inspired a great simplification of our original proof of Theorem 2.

1. A Presentation Matrix of B_L. Suppose a regular projection of the tame link $L \subseteq S^3$, of $\mu \geq 1$ components, is given. If the projection is normalized by removing short arcs surrounding the underpassing point of each crossing, then it consists of pairwise disjoint, tame, simple arcs in the plane. We may denote these arcs e_{ij} ($1 \leq i \leq \mu$ and $1 \leq j \leq j_i$, the latter considered modulo j_i), the indices being chosen so that for each i $e_{i1} \cup \cdots \cup e_{ij_i}$ is the image of the i^{th} component K_i in the projection, and so that $e_{i1}, e_{i2}, \ldots, e_{ij_i}$ appear consecutively around K_i; we orient K_i so that this direction around it is preferred. We refer to the crossing that separates e_{ij} from e_{ij+1} as the ij^{th} crossing of the projection, and we define $\delta_{ij} = 1$ or -1 according to whether the overpassing arc of the ij^{th} crossing is oriented from left to right or from right to left, relative to the orientation of the underpassing component.

As is well known, a presentation $\langle x_{ij};\, r_{ij}\rangle$ of $G = \pi_1(S^3 - L)$ may be obtained from this regular projection. A generator x_{ij} corresponds to each arc e_{ij}, and a relator r_{ij} to each crossing; if e_{pq} is the overpassing arc of the ij^{th} crossing then

$$r_{ij} = x_{pq}^{\delta_{ij}} x_{ij} x_{pq}^{-\delta_{ij}} x_{ij+1}^{-1}.$$

It suits our purpose here to modify this presentation. We introduce new generators y_{ij}, one corresponding to each arc e_{ij} in the projection, and for each such generator we introduce a relator $y_{ij} x_{ij}^{-1} x_{i1}$. (In particular, note that y_{i1} is a relator for each i, so the generators y_{i1} could simply be deleted; in order to keep our description of the presentation as simple as possible, though, we shall not delete them.) Then we delete each generator x_{ij} and each relator $y_{ij} x_{ij}^{-1} x_{i1}$ for $1 \leq i \leq \mu$ and $2 \leq j \leq j_i$, and replace every occurence of a generator x_{ij} in one of the remaining relators by $x_{i1} y_{ij}$. The result is a presentation $\langle x_{i1}, y_{ij};\, y_{i1}, q_{ij}\rangle$ in which there is a generator x_{i1} and a relator y_{i1} for each $i \in \{1, \ldots, \mu\}$, a generator y_{ij} corresponding to each arc e_{ij}, and a relator

$$q_{ij} = (x_{p1} y_{pq})^{\delta_{ij}} x_{i1} y_{ij} (x_{p1} y_{pq})^{-\delta_{ij}} y_{ij+1}^{-1} x_{i1}^{-1}$$

whenever e_{pq} is the overpassing arc in the ij^{th} crossing.

If F is the free group on the set $\{x_{i1}, y_{ij} \mid 1 \leq i \leq \mu,\ 1 \leq j \leq j_i\}$ of generators, then there is an epimorphism $\eta\colon F \to G$ whose kernel is the normal subgroup of F

generatred by $\{y_{i1}\} \cup \{q_{ij}\}$. If α: $G \to H$ is the abelianizing homomorphism, then it is a simple matter to show that $\alpha\eta(y_{ij})=1$ whenever $1 \leq i \leq \mu$ and $1 \leq j \leq j_i$, and that H is the free abelian group on the set of elements $t_i=\alpha\eta(x_{i1})$, $1 \leq i \leq \mu$.

The *Alexander matrix* M of the presentation $\langle x_{i1}, y_{ij}; y_{i1}, q_{ij}\rangle$ is an $(m+\mu)\times(n+\mu)$ matrix, where m is the number of crossings in the projection and $n=\Sigma j_i$ is the number of arcs. The matrix has one row for each relator in the presentation, and one column for each generator; the common entry of the row corresponding to the relator p and the column corresponding to the generator z is $\alpha\eta(\partial(p)/\partial z)$. We order the rows and columns of M in the obvious way: the first μ columns are those corresponding to $x_{11},\ldots, x_{\mu 1}$, and of the remaining n columns the one corresponding to y_{ab} precedes the one corresponding to y_{cd} if $a<c$ or $a=c$ and $b<d$; similarly, the first μ rows are those corresponding to $y_{11},\ldots, y_{\mu 1}$, and of the remaining m rows the one corresponding to q_{ab} precedes the one corresponding to q_{cd} if $a<c$ or $a=c$ and $b<d$. That the Alexander matrix M is a presentation matrix for the Alexander module A_L is well known [2, §3]. Regarding terminology, we should note that Rolfsen [6] uses "Alexander matrix" to refer to an arbitrary presentation matrix of the Alexander invariant B_L, in contrast with our usage here.

Following Crowell and Strauss [3], we define the matrices $N_2(\mu)$ and $N_3(\mu)$. If $\mu \geq 2$, $N_2(\mu)$ is a $\binom{\mu}{2}\times\mu$ matrix, whose columns are indexed by $\{1,\ldots,\mu\}$ and whose rows are indexed by $\{(p, q)|1 \leq p<q \leq \mu\}$; the only nonzero entries in the (p, q) row are $1-t_q$ (in the p^{th} column) and t_p-1 (in the q^{th} column). If $\mu \geq 3$, $N_3(\mu)$ is a $\binom{\mu}{3}\times\binom{\mu}{2}$ matrix, whose columns are indexed by $\{(p, q)|1 \leq p<q \leq \mu\}$ and whose rows are indexed by $\{(p, q, r)|1 \leq p<q<r \leq \mu\}$; the only nonzero entries in the (p, q, r) row are t_r-1 (in the (p, q) column), $1-t_q$ (in the (p, r) column), and t_p-1 (in the (q, r) column). We also adopt specific orderings of the rows and columns of these matrices: $\{1,\ldots,\mu\}$ is ordered in the usual way, $\{(p, q)\}$ is ordered in such a way that (p, q) precedes (p', q') iff $q<q'$ or $q=q'$ and $p<p'$, and $\{(p, q, r)\}$ is ordered so that (p, q, r) precedes (p', q', r') iff $r<r'$ or $r=r'$ and (p, q) precedes (p', q').

It is convenient to partition the Alexander matrix M as $M=(M_1\ M_2)$, where M_1 consists of the first μ columns of M, and M_2 the remaining n. If $\mu \geq 2$, M_1 factors as a product $M_1=M'\cdot N_2(\mu)$ for some $(m+\mu)\times\binom{\mu}{2}$ matrix M'; such a matrix is explicitly described below. As in [3, §6], if $\mu \geq 3$ the matrix

$$P=\begin{pmatrix} M' & M_2 \\ N_3(\mu) & 0 \end{pmatrix}$$

is a presentation matrix for B_L, while if $\mu=2$

$$P=(M' \quad M_2)$$

is a presentation matrix for B_L. If $\mu=1$ then $ZH \cong IH$ as ZH-modules, so the link module sequence of L must split; it is not difficult to deduce from this and the detailed description of the sequence in [2] that then

$$P = M_2$$

is a presentation matrix for B_L.

In order to explicitly describe the matrices M_2 and M', it is convenient to assume that the given regular projection of L contains no crossing which is trivial in the sense that the overpassing arc of the crossing coincides with one of the underpassing arcs; clearly, any such crossing could be removed from the projection. Also, we assume that each j_i is at least two.[1]

Given these assumptions, the $(m+\mu)\times n$ matrix M_2, which has a row for each y_{i1}, a row for each q_{ij}, and a column for each y_{ij}, has these entries: if $1\leq i\leq\mu$ the only nonzero entry in the row corresponding to y_{i1} is a 1 in the column corresponding to y_{i1}; and if e_{pq} is the overpassing arc in the ij^{th} crossing of the projection, then the only nonzero entries in the row corresponding to q_{ij} are $t_i t_p^{\delta_{ij}}$ in the column corresponding to y_{ij}, $-t_i$ in the column corresponding to y_{ij+1}, and either $t_p(1-t_i)$ or t_i-1 in the column corresponding to y_{pq}, according to whether δ_{ij} is 1 or -1.

The $(m+\mu)\times\binom{\mu}{2}$ matrix M' has one row for each y_{i1}, one row for each q_{ij}, and one column for each pair (p, q) with $1\leq p<q\leq\mu$. The rows corresponding to the y_{i1} are without nonzero entries. Suppose e_{pq} is the overcrossing arc in the ij^{th} crossing of the projection. If $p<i$, then the row of M' corresponding to q_{ij} has a single nonzero entry, 1 or $-t_p^{-1}$ (according to whether $\delta_{ij}=1$ or -1), in the (p, i) column. If $p>i$, the row corresponding to q_{ij} has a lone nonzero entry, -1 or t_p^{-1} (according to whether δ_{ij} is 1 or -1), in the (i, p) column. If $p=i$, every entry of the row of M' corresponding to q_{ij} is zero.

We recall that if C is a $c\times d$ matrix with entries in ZH then its *determinantal ideals* are the ideals of ZH given by: $E_k(C)=0$ whenever $k<0$ or $k<d-c$, $E_k(C)=ZH$ whenever $k\geq d$, and if $0<d-c\leq k<d$ then $E_k(C)$ is the ideal generated by the determinants of the $(d-k)\times(d-k)$ submatrices of C. The determinantal ideals of a finitely presented ZH-module are those of any of its presentation matrices; they are independent of the choice of a particular presentation matrix [5, §3.1].

2. A Lemma. In this section we calculate the determinantal ideals of a certain matrix V, which will appear in the proof of Theorem 1 (§3). The calculation depends on a portion of the theory of determinants due to Crowell and

(1) It is not difficult to show that any link, even the trivial link of one component, has some regular projection in the plane satisfying these two assumptions. Furthermore, note that for such a projection $m=n$.

Strauss [3], which we proceed to discuss.

Let R be an integral domain. If $r, s \geq 1$ we use $M_{r,s}(R)$ to denote the R-module consisting of all $r \times s$ matrices with entries in R; also, if $t \geq 1$ then $M_{r,s}(R)^t$ denotes the t-fold Cartesian product of this module with itself. If $C \in M_{c,d}(R)$ then for any $k \geq 0$ a multilinear mapping $g_C: M_{1,c+k}(R)^{d+k} \to R$ can be defined by

$$g_C(U_1,\ldots, U_{d+k}) = \det\left(\begin{pmatrix} U_1 \\ \vdots \\ U_{d+k} \end{pmatrix} \cdot \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\right),$$

where I is a $k \times k$ identity matrix; if $k=0$ the matrix on the right should be C. If $k>c$ and $k \geq d$ another multilinear mapping $f_C: M_{1,k}(R)^{k-c} \to R$ is given by

$$f_C(U_1,\ldots, U_{k-c}) = \det\begin{Bmatrix} U_1 \\ \vdots \\ U_{k-c} \\ C \quad 0 \end{Bmatrix};$$

if $k=d$ the block of zeroes in the lower right-hand corner should not appear.

Suppose $d>c\geq 1$, $C \in M_{c,d}(R)$, and $D \in M_{d,d-c}(R)$. Let $N(C)$ be the collection of all the $c \times c$ submatrices of C, and let $N(D)$ be the collection of all the $(d-c)\times(d-c)$ submatrices of D. We define a bijection $\tau: N(C) \to N(D)$ as follows. If $X \in N(C)$, X can be obtained from C by deleting, say, the $i_1^{\text{th}},\ldots, i_{d-c}^{\text{th}}$ columns of C; $\tau(X)$ is the $(d-c)\times(d-c)$ submatrix of D obtained by deleting all its rows except for the $i_1^{\text{th}},\ldots, i_{d-c}^{\text{th}}$.

PROPOSITION (2.1): *Let C and D be as described. Suppose further that $CD=0$, $E_{d-c}(C)\neq 0$, and $E_0(D)\neq 0$. Then for any $k\geq 0$, and any $X \in N(C)$ with $\det\tau(X)\neq 0$, the multilinear mappings*

$$f_C,\ g_D: M_{1,d+k}(R)^{k+d-c} \longrightarrow R$$

have the property that $(\det X)\cdot g_D = \pm(\det\tau(X))\cdot f_C$ *identically.*

PROOF: First we show that f_C and g_D have the property that $g_D(U_1,\ldots, U_{k+d-c})=0$ whenever $f_C(U_1,\ldots, U_{k+d-c})=0$. For if

$$\det\begin{Bmatrix} U_1 \\ \vdots \\ U_{k+d-c} \\ C \quad 0 \end{Bmatrix} = 0$$

then the rows of this matrix must be dependent, that is, if $C_1',\ldots, C_c'$ are its last c rows then

$$\sum a_i U_i + \sum b_i C_i' = 0$$

for some $a_1,\dots, a_{k+d-c}, b_1,\dots, b_c \in R$, not all of which are zero. Since $E_{d-c}(C)\neq 0$, the last c rows of this matrix are independent, so some of the a_i must be nonzero.

$$0=\sum a_i U_i\cdot\begin{pmatrix} D & 0\\ 0 & I\end{pmatrix}+\sum b_i C_i'\begin{pmatrix} D & 0\\ 0 & I\end{pmatrix}=\sum a_i U_i\cdot\begin{pmatrix} D & 0\\ 0 & I\end{pmatrix},$$

so the rows of the matrix

$$\begin{pmatrix} U_1\\ \vdots\\ U_{k+d-c}\end{pmatrix}\cdot\begin{pmatrix} D & 0\\ 0 & I\end{pmatrix}$$

Then are dependent, so $g_D(U_1,\dots, U_{k+d-c})=0$.

Now, suppose that $X\in N(C)$ is obtained from C by deleting its $i_1^{\text{th}},\dots, i_{d-c}^{\text{th}}$ columns. Choose $U_1,\dots, U_{k+d-c}\in M_{1,d+k}(R)$ so that for $1\leq i\leq k$ the only nonzero entry of U_i is its $(d+i)^{\text{th}}$, which is a 1, and for $1\leq j\leq d-c$ the only nonzero entry of U_{k+j} is its i_j^{th}, which is a 1. A simple expansion by minors shows that $f_C(U_1,\dots, U_{k+d-c})=\pm\det X$; also, it is not difficult to show that $g_D(U_1,\dots, U_{k+d-c})=\pm\det \tau(X)$. In particular, since $\det\tau(X)\neq 0$ it follows from the first paragraph of this proof that $\det X\neq 0$.

Thus we have

$$rf_C(U_1,\dots, U_{k+d-c}) = \pm\, sg_D(U_1,\dots, U_{k+d-c})\neq 0,$$

with $r=\det\tau(X)$ and $s=\det X$. It follows from [3, Theorem (4.1)] that $r\cdot f_C=\pm s\cdot g_D$ identically, as claimed. Q.E.D.

Suppose $L\subseteq S^3$ is a tame link of $\mu\geq 2$ components, given with a regular projection in the plane, as described in Section 1. We define a $(1+j_\mu+\binom{\mu-1}{2})\times(\mu-1+j_\mu)$ matrix V, with entries in ZH_μ, as follows. The first row corresponds to the relator $y_{\mu 1}$, the next rows correspond to the relators $q_{\mu 1},\dots, q_{\mu j_\mu}$, and the last $\binom{\mu-1}{2}$ rows correspond to the triples (p, q, μ) with $1\leq p<q<\mu$. The first $\mu-1$ columns of V correspond to the pairs (p,μ), $1\leq p<\mu$, and the rest correspond to the generators $y_{\mu 1},\dots, y_{\mu j_\mu}$. The only nonzero entry of the first row is a 1 in the column corresponding to $y_{\mu 1}$. For $1\leq j\leq j_\mu$ the row corresponding to $q_{\mu j}$ has these nonzero entries: if e_{pq} is the overpassing arc of the μj^{th} crossing in the projection and $1\leq p<\mu$, then there is an entry of 1 or $-t_p^{-1}$ (according to whether $\delta_{\mu j}=1$ or -1) in the (p,μ) column, an entry of $t_p^{\delta_{\mu j}}$ in the column corresponding to $y_{\mu j}$, and an entry of -1 in the column corresponding to $y_{\mu j+1}$; if $e_{\mu q}$ is the overpassing arc of the μj^{th} crossing, then there is an entry of 1 in the column corresponding to $y_{\mu j}$, and one of -1 in the column corresponding to $y_{\mu j+1}$. Finally, if $1\leq p<q<\mu$ the row of V corresponding to the triple (p, q, μ) has two nonzero entries: $1-t_q$ (in the (p,μ) column) and t_p-1 (in the (q,μ) column).

That is, V is the matrix

$$\begin{array}{cccccccc} (1,\mu) & \cdots & (\mu-1,\mu) & y_{\mu 1} & y_{\mu 2} & \cdots & & y_{\mu j_\mu} \end{array}$$

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & & 0 \\ v_{11} & \cdots & v_{1\mu-1} & v_1 & -1 & & & \\ \cdot & & \cdot & \cdot & & \cdot & & 0 \\ \cdot & & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & & \cdot & & \cdot \\ \cdot & & \cdot & & 0 & & \cdot & -1 \\ v_{j_\mu 1} & \cdots & v_{j_\mu \mu-1} & -1 & & & & v_{j_\mu} \\ & N_2(\mu-1) & & & & 0 & & \end{pmatrix}$$

where the entries v_j and v_{ij} are as described above. (If $\mu=2$, $N_2(\mu-1)$ and the lowermost block of zeroes should not appear.) By adding suitable multiples of the preceding rows to the j^{th} row, $2\leq j\leq j_\mu+1$, and reversing the sign of each of the last $j_\mu-1$ columns of the resulting matrix, we obtain a matrix

$$\begin{pmatrix} W & I \\ X & 0 \end{pmatrix},$$

where

$$X=\begin{pmatrix} x_1 \cdots x_{\mu-1} \\ N_2(\mu-1) \end{pmatrix}$$

and I is a $j_\mu \times j_\mu$ identity matrix. The entries of the first row of X are given by

$$x_i = v_{j_\mu i} + \sum_{j=1}^{j_\mu-1} \Big(\prod_{k=j+1}^{j_\mu} v_k\Big) v_{ji}.$$

It is not difficult, because of the manner in which X was obtained from V, to show that $E_k(V)=E_k(X)\,\forall k\in \boldsymbol{Z}$. (See [5, Theorem 1.4 and Lemma 1.2]; note that our notation differs somewhat from that used there.) As the first step in the calculation of these ideals, we have

LEMMA (2.2):

$$\sum_{i=1}^{\mu-1} x_i\cdot(t_i-1) = \Big(\prod_{i=1}^{\mu-1} t_i^{\ell_i}\Big) - 1$$

and if $1\leq i_1<\cdots<i_k<\mu$

$$\sum_{h=1}^{k} x_{i_h}\cdot(t_{i_h}-1) - \prod_{h=1}^{k} t_{i_h}^{\ell_{i_h}} + 1 \in (IH_\mu)^2.$$

PROOF: Note that whenever $1 \leq j \leq j_\mu$

$$\sum_{i=1}^{\mu-1} v_{ji}\cdot(t_i-1) = v_j - 1,$$

and hence

$$\sum_{i=1}^{\mu-1} x_i\cdot(t_i-1) = (\prod_{j=1}^{j_\mu} v_j) - 1 = (\prod_{i=1}^{\mu-1} t_i^{\ell_i}) - 1.$$

If $1 \leq i < \mu$ and $1 \leq j \leq j_\mu$ then $v_{ji}\cdot(t_i-1)$ is either $t_i^{\delta_{\mu j}}-1$ or 0, according to whether or not the overpassing component of the μj^{th} crossing of the projection is K_i; since $t_i^r - 1 \equiv r\cdot(t_i-1) \pmod{(IH_\mu)^2}$ for any $r \in \boldsymbol{Z}$, it follows that

$$x_i\cdot(t_i-1) \equiv \sum_{j=1}^{j_\mu} v_{ji}\cdot(t_i-1) \equiv \ell_i\cdot(t_i-1) \equiv t_i^{\ell_i} - 1,$$

the congruences holding modulo $(IH_\mu)^2$. Thus if $1 \leq i_1 < \cdots < i_k < \mu$

$$\sum_{h=1}^{k} x_{i_h}\cdot(t_{i_h}-1) \equiv \sum_{h=1}^{k} (t_{i_h}^{\ell_{i_h}}-1) \equiv (\prod_{h=1}^{k} t_{i_h}^{\ell_{i_h}}) - 1$$

modulo $(IH_\mu)^2$. Q.E.D.

If $\mu=2$, X is a 1×1 matrix with the same determinantal ideals as V, so we immediately obtain

COROLLARY (2.3): *If* $\mu=2$, $E_1(V)=\boldsymbol{Z}H_\mu$ *and*

$$E_0(V) = \left(\frac{t_1^{\ell_1}-1}{t_1-1}\right).$$

If $\mu=3$, X is the 2×2 matrix

$$\begin{pmatrix} x_1 & x_2 \\ 1-t_2 & t_1-1 \end{pmatrix},$$

so since the determinantal ideals of X and V coincide we obtain

COROLLARY (2.4): *If* $\mu=3$, $E_2(V)=\boldsymbol{Z}H_\mu$, $E_1(V)=(\ell_1, \ell_2)+IH_\mu$, *and*

$$E_0(V) = ((t_1^{\ell_1}t_2^{\ell_2})-1).$$

PROOF: The determination of $E_0(V)=E_0(X)$ follows immediately from the first assertion of Lemma (2.2), while that of $E_1(V)=E_1(X)$ follows from the congruence $x_i \equiv \ell_i \pmod{IH_\mu}$, which is a consequence of the second assertion of Lemma (2.2). Q.E.D.

The determination of the determinantal ideals of X and V for $\mu \geq 4$ requires a detailed discussion of the submatrices of $N_2(\mu-1)$. To facilitate this discussion it is convenient to introduce a new piece of notation: if $1 \leq i_1 < \cdots < i_k < \mu$ then $N_2(i_1,\ldots, i_k)$ is the submatrix of $N_2(\mu-1)$ obtained by deleting its i^{th} column whenever $i \notin \{i_1,\ldots, i_k\}$, and deleting the row corresponding to (p, q) whenever either of p, q is not among $i_1,\ldots, i_k$. (Equivalently, $N_2(i_1,\ldots, i_k)$ is obtained from $N_2(k)$ by replacing t_j by t_{i_j}, $1 \leq j \leq k$.)

LEMMA (2.5): *Suppose Y is a square submatrix of $N_2(\mu-1)$ and* $\det Y \neq 0$. *Then by permuting rows and columns Y can be brought into upper triangular form.*

PROOF: Suppose Y has y columns and rows. If $y=1$, the assertion of the lemma is trivial.

Proceeding inductively, suppose $y>1$, and suppose Y involves the $i_1^{\text{th}},\ldots, i_y^{\text{th}}$ columns of $N_2(\mu-1)$, where $1 \leq i_1 < \cdots < i_y$. If some row of Y has only one nonzero entry, then by *permuting rows and columns* we can move this entry to the bottom right-hand corner of Y; then the $(y-1)\times(y-1)$ submatrix of Y obtained by deleting its last row and column must have nonzero determinant, soby inductive hypothesis it (and therefore Y) can be put in upper triangular form. On the other hand, if every row of Y has at least two (and, therefore, preciseley two) nonzero entries, then Y is a submatrix of $N_2(i_1,\ldots, i_y)$; however, the columns of $N_2(i_1,\ldots, i_y)$ are linearly dependent (if one multiplies the j^{th} column by $t_{i_j}-1$, $1 \leq j \leq y$, the resulting columns add up to 0), contradicting the assumption that $\det Y \neq 0$.

Q. E. D.

Given a (not necessarily square) submatrix Y of $N_2(\mu-1)$, we define integers $\gamma_1(Y),\ldots, \gamma_{\mu-1}(Y)$, $\rho_1(Y),\ldots, \rho_{\mu-1}(Y)$ as follows: $\gamma_i(Y)$ is 1 or 0 according to whether or not Y involves the i^{th} column of $N_2(\mu-1)$, and $\rho_i(Y)$ is the number of pairs (p, q) such that Y involves the (p, q) row of $N_2(\mu-1)$ and one of p, q is i. These integers are intimately related to the determinants of the square submatrices of $N_2(\mu-1)$, as we see in

LEMMA (2.6): *If Y is a $y\times y$ submatrix of $N_2(\mu-1)$ with nonzero determinant, then*

$$\det Y = \pm \prod_{i=1}^{\mu-1} (t_i-1)^{\rho_i(Y)-\gamma_i(Y)}$$

and, in particular, $\rho_i(Y) \geq \gamma_i(Y)$ for each i.

PROOF: If $y=1$ then there are p, q such that $\gamma_p(Y)=1$, $\gamma_i(Y)=0$ whenever $p \neq i$, $\rho_p(Y)=1=\rho_q(Y)$, and $\rho_i(Y)=0$ whenever $p \neq i \neq q$ (i.e., Y involves the

p^{th} column and either the (p, q) or the (q, p) row of $N_2(\mu-1)$). Then $\det Y = \pm(t_q-1)$, as claimed.

Proceeding inductively, suppose $y>1$. By Lemma (2.5), some row of Y must have only one nonzero entry; let Y_1 be the 1×1 matrix consisting of this entry, and Y_2 the $(y-1)\times(y-1)$ submatrix of Y obtained by deleting the row and column of Y containing this entry; clearly then $\gamma_i(Y)=\gamma_i(Y_1)+\gamma_i(Y_2)$ and $\rho_i(Y)=\rho_i(Y_1)+\rho_i(Y_2)$ for any i. Since $\det Y = \pm(\det Y_1)\cdot(\det Y_2)$, the result now follows from the inductive hypothesis. Q.E.D.

Let $N_1(\mu-1)$ be the $(\mu-1)\times 1$ matrix whose i^{th} entry is t_i-1. Note that $N_2(\mu-1)\cdot N_1(\mu-1)=0$ for any $\mu\geq 3$.

PROPOSITION (2.7): *Let C be a $(\mu-2)\times(\mu-1)$ submatrix of $N_2(\mu-1)$. If $E_1(C)=0$ then f_C is identically zero, while if $E_1(C)\neq 0$*

$$f_C = \pm\left(\prod_{i=1}^{\mu-1}(t_i-1)^{\rho_i(C)-\gamma_i(C)}\right)\cdot g_{N_1(\mu-1)}$$

identically. Conversely, whenever $p_1,\ldots,p_{\mu-1}$ are non-negative integers with $\sum p_i=\mu-3$, $N_2(\mu-1)$ has some $(\mu-2)\times(\mu-1)$ submatrix C such that

$$f_C = \pm\left(\prod_{i=1}^{\mu-1}(t_i-1)^{p_i}\right)\cdot g_{N_1(\mu-1)}$$

identically.

PROOF: First, suppose C is a $(\mu-2)\times(\mu-1)$ submatrix of $N_2(\mu-1)$. If $E_1(C)=0$ then the rows of C must be linearly dependent; clearly then f_C is indeed identically zero.

If $E_1(C)\neq 0$, let Y be a $(\mu-2)\times(\mu-2)$ submatrix of C with nonzero determinant; then Y is obtained from C by deleting a single column, say, the q^{th}. By Lemma (2.6),

$$\det Y = \pm(t_q-1)\cdot\prod_{i=1}^{\mu-1}(t_i-1)^{\rho_i(C)-\gamma_i(C)}.$$

Since $N_2(\mu-1)\cdot N_1(\mu-1)=0$, certainly $C\cdot N_1(\mu-1)=0$. By Proposition (2.1), then,

$$(\det Y)\cdot g_{N_1(\mu-1)} = \pm(\det \tau(Y))\cdot f_C$$

identically, where $\tau(Y)$ is the submatrix of $N_1(\mu-1)$ obtained by deleting all its rows except the q^{th}; that is, $\tau(Y)$ is the 1×1 matrix whose sole entry is t_q-1. That

$$f_C = \pm\left(\prod_{i=1}^{\mu-1}(t_i-1)^{\rho_i(C)-\gamma_i(C)}\right)\cdot g_{N_1(\mu-1)}$$

follows immediately, by cancellation.

For the converse, suppose $p_1, \ldots, p_{\mu-1}$ are as in the statement. As shown by Crowell and Strauss [3, Proposition (5.2)], $N_2(\mu-1)$ has some $(\mu-2)\times(\mu-2)$ submatrix Y with

$$\det Y = \pm(t_1-1)\cdot \prod_{i=1}^{\mu-1} (t_i-1)^{p_i},$$

which does not involve the first column of $N_2(\mu-1)$. If C is the $(\mu-2)\times(\mu-1)$ submatrix of $N_2(\mu-1)$ of which Y is a submatrix, then by Proposition (2.1)

$$(\det Y)\cdot g_{N_1(\mu-1)} = \pm (\det \tau(Y))\cdot f_C$$

identically. Since $\tau(Y)$ is the 1×1 matrix whose lone entry is t_1-1, we obtain

$$f_C = \pm \left(\prod_{i=1}^{\mu-1} (t_i-1)^{p_i}\right)\cdot g_{N_1(\mu-1)}$$

by cancellation. Q. E. D.

Using this result, we can prove

THEOREM (2.8): *If $\mu \geq 4$ then*

$$E_0(V) = \left(\left(\prod_{j=1}^{\mu-1} t_j^{\ell_j}\right)-1\right)\cdot (IH_\mu)^{\mu-3},$$

for $1\leq k\leq \mu-2$

$$E_k(V) \supseteq (IH_\mu)^{\mu-1-k} + \sum_{j=1}^{\mu-1} (\ell_j)\cdot(\{t_r-1 \mid r\neq j\})^{\mu-2-k}$$

$$+ \sum_{p=2}^{\mu-1-k} \Sigma\left(\left(\prod_{q=1}^{p} t_{j_q}^{\ell_{j_q}}\right)-1\right)\cdot(\{t_{j_q}-1\})^{p-2}\cdot(\{t_r-1 \mid r\neq j_q \,\forall q\})^{\mu-1-k-p}$$

where the sum Σ is taken over the set of all p-tuples $(j_1, \ldots, j_p)$ with $1\leq j_1<\cdots< j_p<\mu$, and

$$E_{\mu-1}(V) = ZH_\mu.$$

PROOF: Since X and V have the same determinantal ideals, we may confine our attention to X; that $E_{\mu-1}(X) = ZH_\mu$ follows from the fact that X has only $\mu-1$ columns.

The ideal $E_0(X)$ is generated by the determinants of the $(\mu-1)\times(\mu-1)$ submatrices of X. Since the columns of $N_2(\mu-1)$ are linearly dependent, if such a submatrix is to have nonzero determinant it must involve the first row X_1 of X. Thus $E_0(X)$ is the ideal of ZH_μ generated by the values of $f_C(X_1)$, as C varies over the set of $(\mu-2)\times(\mu-1)$ submatrices of $N_2(\mu-1)$; by Proposition (2.7), then,

$$E_0(X) = (g_{N_1(\mu-1)}(X_1)) \cdot (IH_\mu)^{\mu-3}.$$

The determinantion of $E_0(X)$ is completed by the calculation (in the first assertion of Lemma (2.2)) of $g_{N_1(\mu-1)}(X_1)$.

Suppose $1 \le k \le \mu-2$, and let E be the ideal which, according to the statement, is contained in $E_k(V)=E_k(X)$.

Recall that by [3, Proposition (5.2)], $(IH_\mu)^{\mu-1-k}=E_k(N_2(\mu-1))$, so since $N_2(\mu-1)$ is a submatrix of X, $(IH_\mu)^{\mu-1-k} \subseteq E_k(X)$.

If $1 \le j \le \mu-1$, then by [3, Proposition (5.2)] $(\{t_r-1 | r \ne j\})^{\mu-2-k} = E_k(N_2(1,\dots,j-1,j+1,\dots,\mu-1))$; clearly $(x_j)\cdot E_k(N_2(1,\dots,j-1,j+1,\dots,\mu-1)) \subseteq E_k(X)$. By the second assertion of Lemma (2.2) $x_j \equiv \ell_j \pmod{IH_\mu}$, so since $(IH_\mu)^{\mu-1-k} \subseteq E_k(V)$, $(\ell_j)\cdot(\{t_r-1 | r \ne j\})^{\mu-2-k} \subseteq E_k(V)$.

Now, suppose that $2 \le p \le \mu-1-k$, $1 \le j_1 < \cdots < j_p < \mu$, $\{r_1,\dots,r_{\mu-1-p}\} = \{1,\dots,\mu-1\}-\{j_1,\dots,j_p\}$, and $y_1,\dots,y_p$, $z_1,\dots,z_{\mu-1-p}$ are non-negative integers with $\sum y_q = p-2$ and $\sum z_q = \mu-1-k-p$. By [3, Proposition (5.2)] $N_2(r_1,\dots,r_{\mu-1-p})$ has a $(\mu-1-k-p)\times(\mu-1-k-p)$ submatrix Y' with

$$\det Y' = \pm \prod_{q=1}^{\mu-1-p} (t_{r_q}-1)^{z_q}.$$

By the first part of this proof, the matrix

$$\begin{pmatrix} x_{j_1} \cdots x_{j_p} \\ N_2(j_1,\dots,j_p) \end{pmatrix}$$

has a $p \times p$ submatrix Y'' with

$$\det Y'' = \pm\, g_{N_1(j_1,\dots,j_p)}((x_{j_1} \cdots x_{j_p})) \cdot \prod_{q=1}^{p} (t_{j_q}-1)^{y_q}.$$

It follows from the second assertion of Lemma (2.2) that

$$\det Y'' \equiv \pm\Big(\Big(\prod_{q=1}^{p} t_{j_q}^{\ell_{j_q}} - 1\Big) \cdot \prod_{q=1}^{p} (t_{j_q}-1)^{y_q} \pmod{(IH_\mu)^p}.$$

The matrix X has a $(\mu-1-k)\times(\mu-1-k)$ submatrix Y which, when its rows and columns are suitably permuted, is of the form

$$\begin{pmatrix} Y'' & Z \\ 0 & Y' \end{pmatrix}$$

for some matrix Z. Hence

$$\begin{aligned} \det Y &= \pm (\det Y') \cdot (\det Y'') \\ &\equiv \pm\Big(\Big(\prod_{q=1}^{p} t_{j_q}^{\ell_{j_q}}\Big) - 1\Big) \Big(\prod_{q=1}^{p} (t_{j_q}-1)^{y_q}\Big) \Big(\prod_{q=1}^{\mu-1-p} (t_{r_q}-1)^{z_q}\Big) \end{aligned}$$

modulo $(IH_\mu)^{\mu-1-k}$, so since $(IH_\mu)^{\mu-1-k} \subseteq E_k(V)$ this product is an element of $E_k(V)$.

This completes the proof that $E \subseteq E_k(V)$. Q. E. D.

The inclusion of Theorem (2.8) is actually an equality, but since this fact will not be needed, we will not verify it.

3. Proof of Theorem 1. Let $L \subseteq S^3$ be a tame link of $\mu \geq 2$ components with a regular projection in the plane, and let P be the $\left(\mu+m+\binom{\mu}{3}\right)\times\left(\binom{\mu}{2}+n\right)$ presentation matrix of B_L described in Section 1. Then $E_k(B_L)=E_k(P)$ for any value of k, so if ϕ: $ZH \to ZH_\mu$ is the homomorphism defined in the introduction $\phi E_k(B_L)=\phi E_k(P)=E_k(\phi(P))$ for any k, where $\phi(P)$ is the matrix whose entries are the images under ϕ of those of P.

We rearrange the rows and columns of P in the following manner, obtaining thereby a matrix Q with the same determinantal ideals as P. The first $\binom{\mu-1}{2}$ columns of Q are the columns of P corresponding to the pairs (p, q), $1 \leq p < q < \mu$, and its next $n-j_\mu$ columns are the columns of P corresponding to the generators y_{ij}, $1 \leq i < \mu$ and $1 \leq j \leq j_i$; the next $\mu-1$ columns of Q are those of P corresponding to the pairs (p, μ), $1 \leq p < \mu$, and the last j_μ columns of Q are the columns of P corresponding to the generators $y_{\mu j}$, $1 \leq j \leq j_\mu$. The first $\mu-1$ rows of Q are those of P corresponding to the relators y_{i1}, $1 \leq i < \mu$, the next $m-j_\mu$ rows of Q are those of P corresponding to the relators q_{ij}, $1 \leq i < \mu$ and $1 \leq j \leq j_i$, and its next $\binom{\mu-1}{3}$ rows are the rows of P corresponding to the triples (p, q, r), $1 \leq p < q < r < \mu$. The next row of Q is the row of P corresponding to the relator $y_{\mu 1}$, and after that come the rows of P corresponding to the relators $q_{\mu j}$, $1 \leq j \leq j_\mu$; the last $\binom{\mu-1}{2}$ rows of Q are the rows of P corresponding to the triples (p, q, μ), $1 \leq p < q < \mu$.

It is not difficult, using the explicit description of the entries of P given in Section 1, to ascertain that $\phi(Q)$ may be partitioned as

$$\phi(Q) = \begin{pmatrix} P' & U \\ 0 & V \end{pmatrix},$$

where P' is a $\left(\mu-1+m-j_\mu+\binom{\mu-1}{3}\right)\times\left(\binom{\mu-1}{2}+n-j_\mu\right)$ matrix, and V is the $\left(1+j_\mu+\binom{\mu-1}{2}\right)\times(\mu-1+j_\mu)$ matrix discussed in Section 2.

LEMMA (3.1): *For any $k \in Z$*

$$E_k(P') \supseteq E_k(\phi(Q)) \supseteq \sum_i E_i(V)E_{k-i}(P').$$

PROOF: The former inclusion follows from the observation that if Y is a

$y\times y$ submatrix of $\phi(Q)$, $y\geq\mu+j_\mu$, then the expansion of det Y by minors along the last $\mu-1+j_\mu$ columns of Y expresses det Y as the sum of certain multiples of the determinants of certain $(y-\mu+1-j_\mu)\times(y-\mu+1-j_\mu)$ submatrices of P'.

The latter inclusion follows from the observation that if Y' is a $y'\times y'$ submatrix of P', and Y'' is a $y''\times y''$ submatrix of V, then $\phi(Q)$ has a $(y'+y'')\times(y'+y'')$ submatrix whose determinant is $(\det Y')\cdot(\det Y'')$. Q. E. D.

Just as in [9, §4], a regular projection of the link L_μ in the plane can be obtained from the given regular projection of L, by replacing each crossing in which K_μ passes over some component of L_μ by a trivial crossing. If this projection of L_μ is used to obtain a presentation matrix for the ZH_μ-module B_{L_μ}, as in Section 1, then P' will be the presentation matrix obtained; hence $E_k(B_{L_\mu})=E_k(P')$ for any value of k.

Since $\phi E_k(B_L)=\phi E_k(P)=\phi E_k(Q)=E_k(\phi(Q))$ for any k, we conclude from this and Lemma (3.1) that

$$E_k(B_{L_\mu})\supseteq\phi E_k(B_L)\supseteq\sum_i E_{\mu-1-i}(V)E_{k-\mu+1+i}(B_{L_\mu})$$

for any $k\in\boldsymbol{Z}$. Theorem 1 now follows from Theorem (2.8).

4. Proof of Theorem 2. Consider the ZH-module $B_L/IH\cdot B_L$. If P is any presentation matrix for B_L (e.g., the one discussed in Section 1) then clearly

$$\begin{pmatrix} P \\ (t_1-1)I \\ \vdots \\ (t_\mu-1)I \end{pmatrix}$$

is a presentation matrix for this module, where I is an identity matrix. From [10, Lemma (3.1)] it follows that

$$\sum_{i\geq 0}E_{k+i}(B_L)\cdot(IH)^i=\sum_{i\geq 0}E_{k+i}(B_L/IH\cdot B_L)\cdot(IH)^i$$

for any $k\in\boldsymbol{Z}$.

Massey [4, Lemma 1] has observed that $B_L/IH\cdot B_L$ and G_2/G_3 are isomorphic abelian groups. Therefore, if we consider the latter as a trivial ZH-module (i.e., one with the property that $h\cdot x=x\,\forall h\in H\,\forall x\in G_2/G_3$), we have $E_k(B_L/IH\cdot B_L)=E_k(G_2/G_3)$, and hence

$$\sum_{i\geq 0}E_{k+i}(B_L)\cdot(IH)^i=\sum_{i\geq 0}E_{k+i}(G_2/G_3)\cdot(IH)^i,$$

for all values of k.

As noted in the introduction, Chen [1] has shown that the matrix λ is a presentation matrix for the abelian group G_2/G_3. It follows that the trivial $\boldsymbol{Z}H$-module G_2/G_3 has the presentation matrix

$$\begin{pmatrix} \lambda \\ (t_1-1)I \\ \vdots \\ (t_\mu-1)I \end{pmatrix},$$

where I is, again, an identity matrix. Another application of [10, Lemma (3.1)] shows that

$$\sum_{i\geq 0} E_{k+i}(B_L)\cdot(IH)^i = \sum_{i\geq 0} E_{k+i}(\lambda)\cdot(IH)^i$$

for any $k\in\boldsymbol{Z}$, and in particular

$$\sum_{i\geq 0} E_{\binom{\mu}{2}+i}(B_L)\cdot(IH)^i = \sum_{i\geq 0} E_{\binom{\mu}{2}+i}(\lambda)\cdot(IH)^i = \boldsymbol{Z}H.$$

(Note that this provides an alternate proof of Corollary 1.) Consequently, if $1\leq k\leq\binom{\mu}{2}$

$$\begin{aligned}\sum_{i=0}^{k-1} & E_{\binom{\mu}{2}-k+i}(B_L)\cdot(IH)^i + (IH)^k \\ &= \sum_{i\geq 0} E_{\binom{\mu}{2}-k+i}(B_L)\cdot(IH)^i \\ &= \sum_{i\geq 0} E_{\binom{\mu}{2}-k+i}(\lambda)\cdot(IH)^i \\ &= \sum_{i=0}^{k-1} E_{\binom{\mu}{2}-k+i}(\lambda)\cdot(IH)^i + (IH)^k.\end{aligned}$$

This completes the proof of Theorem 2.

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