TRALDI, L.
KOBE J.MATH.,
5(1988) 233–264

CONWAY'S POTENTIAL FUNCTION AND ITS TAYLOR SERIES

dedicated to William S. Massey

By Lorenzo TRALDI

Reprinted from KOBE JOURNAL of MATHEMATICS
Vol. 5 No. 2, December, 1988

TRALDI, L.
KOBE J. MATH.,
5(1988) 233–264

CONWAY'S POTENTIAL FUNCTION AND ITS TAYLOR SERIES

dedicated to William S. Massey

By Lorenzo TRALDI

(Received September 1, 1987)

1. Introduction

Recall that a (tame, oriented) *link* in the three-sphere S^3 is the disjoint union $L = K_1 \cup \cdots \cup K_\mu$ of a finite number $\mu \geq 1$ of oriented, polygonal, simple closed curves (its *components*). Two links are said to be *ambient isotopic* (or of the same *link type*) iff there is an orientation-preserving auto-homeomorphism of S^3 which maps one onto the other in such a way that the indices and orientations of their components correspond.

The *group* of a link L is the fundamental group $G = \pi_1(S^3 - L)$ of its complement. By Alexander duality, the abelianization $H = G/G' = H_1(S^3 - L)$ is a free abelian group of rank μ. A basis $\{t_1, \ldots, t_\mu\}$ of this abelian group consists of meridians to the components of L, that is, the homology classes of closed curves $m_1, \ldots, m_\mu$ in $S^3 - L$, where each m_i has linking number + 1 with K_i and is unlinked from every other component of L. The integral group ring $\boldsymbol{Z}H$ can be identified with the ring $\boldsymbol{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ of integral Laurent polynomials in $t_1, \ldots, t_\mu$, in an obvious way. The ring possesses an involution (which we denote by an overbar) given by $t_i \mapsto t_i^{-1}$. Another useful homomorphism is the augmentation map $\epsilon : \boldsymbol{Z}H \to \boldsymbol{Z}$ given by $\epsilon(t_i) = 1 \ \forall i$; its kernel is the augmentation ideal IH of $\boldsymbol{Z}H$.

One of the most popular invariants in the classical theory of links is "the" Alexander polynomial [8, 11, 35, 41] $\Delta_L \in \boldsymbol{Z}H$. This polynomial is not an absolute invariant of link type; only the principal ideal (Δ_L) is invariant. (That is, Δ_L is only invariant up to multiplication by units of $\boldsymbol{Z}H$.) An interesting property of the Alexander polynomial is its symmetry: if $\mu \geq 2$,

$$\overline{\Delta}_L = (-1)^\mu t_1^{n_1} \cdots t_\mu^{n_\mu} \cdot \Delta_L, \tag{1.1}$$

where each n_i is not congruent (modulo 2) to the linking number $\ell(K_i, L - K_i)$ (i.e., the sum of the linking numbers of K_i with the other components of L) [42]; while if $\mu = 1$,

$$\overline{\Delta}_L = t_1^{n_1} \cdot \Delta_L \tag{1.2}$$

for some even integer n_1 [37].

More recently, Conway [5] has introduced his "potential function" ∇_L, an invariant designed to be a normalized version of Δ_L (that is, ∇_L determines Δ_L and is, moreover, an absolute invariant of link type). The two are related by

$$\nabla_L = \pm t_1^{n_1} \cdots t_\mu^{n_\mu} \cdot \Delta_L(t_1^2, \ldots, t_\mu^2) \tag{1.3}$$

if $\mu > 1$, and

$$\nabla_L = \pm t_1^{n_1} \cdot \Delta_L(t_1^2)/(t_1 - t_1^{-1}) \tag{1.4}$$

if $\mu = 1$, where $n_1, \ldots, n_\mu$ are integers chosen so that ∇_L will satisfy the equality $\nabla_L = (-1)^\mu \overline{\nabla}_L$. (That is, for a given choice of Δ_L these $n_1, \ldots, n_\mu$ coincide with those of (1.1) and (1.2).) We actually follow the definition of the potential function given by Hartley [10], which differs slightly from Conway's; see §2.

The following theorems are concerned with the difference between the potential functions of links that are related by equivalence relations weaker than ambient isotopy; we recall the definitions of these relations. Let μS^1 denote the disjoint union of μ copies of the circle S^1. A (topological) *I-equivalence* [39] between two μ-component links L_0 and L_1 is an imbedding

$$i : (\mu S^1) \times [0, 1] \hookrightarrow S^3 \times [0, 1]$$

such that $L_j \times \{j\} = i((\mu S^1) \times [0, 1]) \cap (S^3 \times \{j\}) = i((\mu S^1) \times \{j\})$ for $j \in \{0, 1\}$. In addition, we require that the I-equivalence be compatible with the orientations and indexings of the components of L_0 and L_1. Two special types of piecewise linear I-equivalences are distinguished, the *concordances* (which are locally flat) and the *isotopies* (which are level-preserving, i.e., which have $i((\mu S^1) \times \{t\}) \subseteq S^3 \times \{t\}\ \forall t$). (Whenever we refer to a concordance or isotopy, we have a piecewise linear one in mind.) It is important to note that the relation of piecewise linear I-equivalence is generated by the concordances and isotopies [36].

THEOREM 1. *If* $L_0 = K_{10} \cup \cdots \cup K_{\mu 0}$ *and* $L_1 = K_{11} \cup \cdots \cup K_{\mu 1}$ *are isotopic then*

$$\nabla_{L_0} \cdot \Pi \nabla_{K_{i1}}(t_i) = \nabla_{L_1} \cdot \Pi \nabla_{K_{i0}}(t_i).$$

THEOREM 2. *If* L_0 *and* L_1 *are concordant then there are* $F_0, F_1 \in \mathbf{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ *with* $\epsilon(F_0) = 1 = \epsilon(F_1)$ *and*

$$\nabla_{L_0} \cdot F_0(t_1^2, \ldots, t_\mu^2) \cdot F_0(t_1^{-2}, \ldots, t_\mu^{-2})$$
$$= \nabla_{L_1} \cdot F_1(t_1^2, \ldots, t_\mu^2) \cdot F_1(t_1^{-2}, \ldots, t_\mu^{-2}).$$

THEOREM 3. *If L_0 and L_1 are topologically I-equivalent, then there are $F_0, F_1 \in \boldsymbol{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ with $\epsilon(F_0) = 1 = \epsilon(F_1)$, $F_0 = \overline{F}_0$, $F_1 = \overline{F}_1$, and*

$$\nabla_{L_0} \cdot F_0(t_1^2, \ldots, t_\mu^2) = \pm \nabla_{L_1} \cdot F_1(t_1^2, \ldots, t_\mu^2).$$

None of these results is really original. Theorem 1 follows immediately from Rolfsen's theorem [33] that L_0 and L_1 are isotopic iff there is a link L which can be obtained from L_0 by connected-summing some knots into its components, and can be obtained in the same way from L_1 (generally using different knots). At least up to sign, Thoerem 2 follows from the analogous result for the Alexander polynomial, due to Fox and Milnor [9], Kawauchi [17], and Nakagawa [31] (see also [46]). Theorem 3 also follows from the corresponding result for the Alexander polynomial, due to Massey [24] and Levine [20]. We conjecture that the ± in Theorem 3 is unnecessary, though a general proof eludes us; in §3 we verify this conjecture for I-equivalences that can be extended in a certain way, and in §8 we show that it holds for arbitrary I-equivalences when $\mu = 2$.

A basic idea of this paper is the simple observation that these theorems assert that in each case ∇_{L_0} and ∇_{L_1} differ only by certain types of factors. Hence if we simplify the potential functions of links by deleting the appropriate factors, we obtain quotients of ∇ that are invariants for these equivalence relations (in the case of topological I-equivalence, if our conjecture is incorrect the resulting quotient might only be an invariant up to sign). We present these simplified versions of the potential function in detail in §4.

In §5 we discuss the additional information about the isotopy, concordance, or I-equivalence class of a link that can be obtained by considering the potential functions of its sublinks, in relation to its own.

In §6 we discuss the analogous process of simplifying the reduced potential function D_L by deleting certain of its factors. (Recall that if $\pi\colon \boldsymbol{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}] \to \boldsymbol{Z}[t, t^{-1}]$ is the homomorphism given by $\pi(t_i) = t \ \forall i$ then the reduced potential function is $D_L = \pi(\nabla_L) \cdot (t - t^{-1})$.) A special aspect of the reduced potential function is that it can be written as a polynomial

$$D_L = z^{\mu - 1} \cdot (a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n})$$

in $z = t - t^{-1}$ [14, 16, 27]. We recover and strengthen some results of Cochran [3] on the extent to which the a_i of I-equivalent links can vary, by considering the coefficients in the analogous expressions (as polynomials in z) of the simplified versions of D_L.

Several authors [12, 24, 30, 38, 43] have observed that when $\mu \geq 2$, the expansion of Δ_L as a power series in $t_1 - 1, \ldots, t_\mu - 1$ (i.e., the Taylor series expansion of Δ_L at the point $(1, \ldots, 1)$) is useful and revealing, particularly in connection with the relationship between the Alexander polynomial and Milnor's $\bar{\mu}$-invariants. In §7 we present the corresponding expansion

$$t_1^{q_1} \cdots t_\mu^{q_\mu} \cdot \nabla_L = \Sigma\, c(r_1, \ldots, r_\mu)(t_1^2-1)^{r_1} \cdots (t_\mu^2-1)^{r_\mu}$$

of a multiple of the potential function as a power series in $t_1^2 - 1, \ldots, t_\mu^2 - 1$. (Multiples of the simplified versions of the potential function can also be expanded in this way.) In §8 we discuss the relationship between Milnor's $\bar{\mu}$-invariants and the potential function, and use it to prove results in both directions: using the $\bar{\mu}$-invariants we derive the fact that the ± in Theorem 3 is unnecessary when $\mu = 2$; and by expanding the appropriate simplification of the potential function we show that certain combinations of $\bar{\mu}$-invariants can be "lifted" to integers that are invariants of topological I-equivalence (at least up to sign). The most striking result of this type is that the $\bar{\mu}$-invariants of type $\bar{\mu}(1, \ldots, 1, 2, \ldots, 2)$ can be lifted (without ambiguity of sign) to integer invariants of topological I-equivalence of two-component links. (This particular result is a refinement of a closely related result of Smythe [38], who showed that when $\mu = 2$ and the linking number is nonzero, the Taylor series expansion of Δ_L at the point (1, 1) can be normalized so as to be an invariant of F-isotopy, and that the coefficients in this power series are lifts of these $\bar{\mu}$-invariants.)

In §9 we discuss the application of these techniques to another invariant related to the (reduced) potential function, namely the Arf invariant. First, we show that there is an integer concordance invariant, derived from the appropriate simplification of the reduced potential function, which reduces (mod 2) to the Arf invariant whenever the Arf invariant is well-defined. Second, we prove that for links in which all the linking numbers are even, the Arf invariant is determined by the Arf invariants of the sublinks consisting of three or fewer components.

In §10 we show that for two-component links the coefficient $c(1, 1)$ is (somewhat) related to the gordian number. The most striking instance of this relationship is the fact that a two-component link of linking number 0 and gordian number 1 has $|c(1,1)|$ (or equivalently, $|\bar{\mu}(1,1,2,2)|$) a perfect square.

Before considering our introduction complete, we should mention that the initial impetus for this work was provided by Cochran's paper [3], in which he investigated the extent to which the coefficients $a_0, a_2, \ldots$ in D_L vary under piecewise linear I-equivalence, and also related them to several other invariants, including the $\bar{\mu}$-invariants. We are grateful to him for writing [3], and for his interesting correspondence.

We should also thank the Committee on Advanced Study and Research of Lafayette College, for its support.

Finally, I should take this opportunity to express my personal gratitude to William S. Massey, who introduced me to knots and links. Of course, his paper [24]

had an obvious influence on this work. More importantly, though, his interest and encouragement have been continuing inspirations to me. I am deeply thankful to have had the opportunity to be his student, and to be associated with him since then.

2. The left- and right-handed potential functions

In this section we discuss the relationship between Conway's original "left-handed" potential function [5] and the "right-handed" version introduced by Hartley [10] (which we are using here). The main result (Theorem (2.2)) is of some interest in its own right; also, its proof illustrates a simple technique (which we will use again) of using linking numbers to determine the sign ± in (1.3). (The ± in (1.4) is always chosen so that when $\mu = 1$, $D_L(1) = 1$.)

Torres [41] showed that if $L = K_1 \cup K_2$ and $\ell = \ell(K_1, K_2)$ is the linking number of K_1 and K_2 then $\epsilon(\Delta_L) = \pm\ell$. This result was extended to links of more than two components by Hosokawa [13]. Hartley [10] gives the following version of this result for the potential function. Suppose $\mu \geq 2$, and let $\lambda = (\lambda_{ij})$ be the $\mu \times \mu$ matrix given by

$$\lambda_{ii} = \sum_{j \neq i} \ell(K_i, K_j)$$

and (for $i \neq j$) $\lambda_{ij} = -\ell(K_i, K_j)$. For $i, j \in \{1, \ldots, \mu\}$ let $\lambda^{(ij)}$ be the submatrix obtained by deleting the ith row and jth column of λ. Then in $D_L = z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots)$, a_0 is $(-1)^{i+j} \det \lambda^{(ij)}$ (for every choice of i, j).

Hoste [15] states an analogous result for Conway's original version of the potential function, but it differs from Hartley's by a factor of $(-1)^{\mu-1}$. Both Hartley and Hoste use right-handed linking numbers (as do we), and Conway's potential function also satisfies (1.3); hence when $a_0 \neq 0$ and $\mu \geq 2$, Conway's potential function is $(-1)^{\mu-1}\nabla$.

What can be said when $\mu = 1$ or $a_0 = 0$? In this case we use the version of Torres' second relation [41] appropriate to the potential function, namely: if $\widetilde{L} = L \cup K_{\mu+1}$ then

$$\nabla_{\widetilde{L}}(t_1, \ldots, t_\mu, 1) = ((\Pi\, t_i^{\ell_i}) - (\Pi\, t_i^{-\ell_i})) \cdot \nabla_L ,$$

where for $i \leq \mu$, $\ell_i = \ell(K_i, K_{\mu+1})$ [10]. Conway [5] states the same relation for his potential function, but uses left-handed linking numbers. Consequently, if we happen to know that the two potential functions of $\widetilde{L}$ differ by a factor of $(-1)^\mu$, and at least one ℓ_i is nonzero, we may conclude that the potential functions of L differ by a factor of $(-1)^{\mu-1}$.

LEMMA (2.1). *Let $L = K_1 \cup \cdots \cup K_\mu$ be a link of $\mu \geq 1$ components. Then there is a link $\widetilde{L} = L \cup K_{\mu+1}$ whose a_0 is nonzero.*

PROOF. Consider a link $\widetilde{L} = L \cup K_{\mu+1}$ obtained from L by inserting a new component $K_{\mu+1}$ in such a way that the various linking numbers $\ell(K_i, K_{\mu+1})$, $i \in \{1, \ldots, \mu\}$, coincide; say, $\ell(K_i, K_{\mu+1}) = x \; \forall i \in \{1, \ldots, \mu\}$.

Then the associated matrix $\widetilde{\lambda}$ will have

$$\widetilde{\lambda}^{(\mu+1\ \mu+1)} = \lambda + xI,$$

where I is a $\mu \times \mu$ identity matrix, and hence $\det \widetilde{\lambda}^{(\mu+1\ \mu+1)}$ will be a polynomial in x, of degree μ. Consequently, $\widetilde{\lambda}^{(\mu+1\ \mu+1)}$ will be singular for at most μ different values of x. Choosing x appropriately, then, we can arrange that $\widetilde{L}$ have $\det \widetilde{\lambda}^{(\mu+1\ \mu+1)} \neq 0$. Q.E.D.

Combining Lemma (2.1) with the comments preceding it, we conclude

THEOREM (2.2). *Conway's and Hartley's versions of the potential function always differ only by a factor of* $(-1)^{\mu-1}$.

A very different proof of this theorem has been given by V. G. Turaev (see Remark 4.2.9 of [46]).

3. Theorems 1, 2, and 3

In this section we deduce Theorems 1, 2, and 3 from other results already in the literature.

As we mentioned in the introduction, Theorem 1 follows immediately from Rolfsen's [33] characterization of isotopy in terms of connected sums with knots. For if the connected sum $L \# K$ is obtained by splicing the knot K into the ith component of the link L, then the potential function of $L \# K$ is $\nabla_L \cdot D_K(t_i)$ [5].

LEMMA (3.1). *The ring* $Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ *is a unique factorization domain, in which the units are the various monomials* $t_1^{a_1} \cdots t_\mu^{a_\mu}$ *and their negatives. The irreducibles of* $Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ *are unit multiples of those irreducibles of the polynomial ring* $Z[t_1, \ldots, t_\mu]$ *not among* $t_1, \ldots, t_\mu$.

PROOF. The properties mentioned in the lemma follow readily from the corresponding properties of the ordinary polynomial ring $Z[t_1, \ldots, t_\mu]$ and the fact that every Laurent polynomial is a monomial multiple of an ordinary polynomial. Q.E.D.

If L is a link in S^3 and $p: \widetilde{X} \to X = S^3 - L$ is the universal abelian covering, then $H = G/G'$ acts on $\widetilde{X}$ by covering translations, and this induces a ZH-module structure on the homology group $H_1(\widetilde{X}; Z)$. This ZH-module is the *Alexander invariant* of the link L; it is finitely presentable. Any g.c.d. of its order ideal is an Alexander polynomial of L.

We discuss Theorem 3 before Theorem 2. Suppose L_0 and L_1 are topologically

I-equivalent. According to Massey [24], it follows that the IH-adic completions of their Alexander invariants are isomorphic modules over the IH-adic completion of $\mathbf{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$. Levine [20] remarks that this implies that the localizations of their Alexander invariants at $S = \{ f \in \mathbf{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}] \mid \epsilon(f) = \pm 1\}$ are isomorphic as modules over the localized ring $\mathbf{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]_S$. It follows that the Alexander polynomials of L_0 and L_1 generate the same principal ideal in this localized ring; equivalently, there are $f_0, f_1 \in S$ with $\epsilon(f_0) = 1 = \epsilon(f_1)$ and $f_0 \cdot \Delta_{L_0} = \pm f_1 \cdot \Delta_{L_1}$.

It follows that if either Alexander polynomial is zero, then both are; we may take $f_0 = f_1 = 1$ in this case. Otherwise, we may assume that f_0 and f_1 are in lowest terms (i.e., they have no common irreducible factor). As Δ_{L_0} is an associate of $\overline{\Delta}_{L_0}$, and Δ_{L_1} is an associate of $\overline{\Delta}_{L_1}$ (by (1.1), (1.2)), it follows that then f_0 is an associate of $\overline{f}_0$ and f_1 is an associate of $\overline{f}_1$. By the lemma below, f_0 has an associate F_0 with $F_0 = \overline{F}_0$ and $\epsilon(F_0) = 1$, and similarly for f_1; say $f_0 = u_0 F_0$ and $f_1 = u_1 F_1$, where u_0 and u_1 are monomials. Then we have $F_0 \cdot (u_0 \Delta_{L_0}) = \pm F_1 \cdot (u_1 \Delta_{L_1})$. Since $u_0 \Delta_{L_0}$ is itself an Alexander polynomial for L_0, and $u_1 \Delta_{L_1}$ is an Alexander polynomial for L_1, we conclude from (1.3), (1.4) that

$$F_0(t_1^2, \ldots, t_\mu^2) \cdot \nabla_{L_0} = (\pm u) \cdot F_1(t_1^2, \ldots, t_\mu^2) \cdot \nabla_{L_1}$$

for some monomial u. Since $F_0 = \overline{F}_0$, $F_1 = \overline{F}_1$, $\nabla_{L_0} = (-1)^{\mu-1} \overline{\nabla}_{L_0}$, and $\nabla_{L_1} = (-1)^{\mu-1} \overline{\nabla}_{L_1}$, $\pm u$ must be ± 1. This and the following lemma complete the proof of Theorem 3.

Lemma (3.2). *Suppose $f \in \mathbf{Z}[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ is an associate of $\overline{f}$ and has $\epsilon(f) = 1$. Then there is a unique associate F of f with $\epsilon(F) = 1$ and $F = \overline{F}$.*

Proof. By assumption, there are integers $a_1, \ldots, a_\mu$ with

$$\overline{f} = \pm t_1^{a_1} \cdots t_\mu^{a_\mu} f.$$

Since $\epsilon(f) = \epsilon(\overline{f}) \neq 0$, the $\pm$ must be $+$. To prove the lemma, it suffices to show that $a_1, \ldots, a_\mu$ must all be even.

We prove this first in the case $\mu = 1$. Then we have $\overline{f} = t_1^a f$, so $(t_1^a - 1)f$ is divisible by $t_1 - t_1^{-1}$, or equivalently, by $t_1^2 - 1 = (t_1 + 1)(t_1 - 1)$. Neither $t_1 + 1$ nor $t_1 - 1$ could possibly divide f, since $\epsilon(f)$ is odd, so necessarily $t_1^2 - 1$ divides $t_1^a - 1$; that is, a is even.

Now suppose $\mu > 1$, and let $i \in \{1, \ldots, \mu\}$. Define $f_i = f(1, \ldots, 1, t_i, 1, \ldots, 1)$. Then $f_i(1) = 1$, and $\overline{f}_i = t_i^{a_i} f_i$, so by the first part of the proof a_i is even. Q.E.D.

To discuss Theorem 2 and our conjecture that the $\pm$ in Theorem 3 is unnecessary, we need the following notion. We say that an I-equivalence i between links L_0 and L_1 is *strongly extendable* if there is an I-equivalence $\tilde{i}$ between links $\tilde{L}_0$ and $\tilde{L}_1$ such that $\tilde{i}$ restricts to i (consequently, L_0 and L_1 are corresponding sublinks of $\tilde{L}_0$ and $\tilde{L}_1$) and the coefficient a_0 in $D_{\tilde{L}_0} = z^{\tilde{\mu}-1} \cdot (a_0 + a_2 z^2 + \cdots)$ is nonzero. We say i is *extendable* if it is the composition of finitely many strongly extendable I-equivalences. Note that the extendable I-equivalences determine an equivalence relation on the set of all link types, generated by the strongly extendable I-equivalences.

LEMMA (3.3). *Suppose L is a sublink of a link $\tilde{L}$, and $\tilde{L}$ has $a_0 \neq 0$. Then the components of $\tilde{L} - L$ can be listed as $\tilde{K}_1, \ldots, \tilde{K}_\nu$ in such a way that for each $i \in \{1, \ldots, \nu\}$, $\tilde{K}_i$ has nonzero linking number with at least one component of $\tilde{L} - (\tilde{K}_1 \cup \cdots \cup \tilde{K}_{i-1})$.*

PROOF. Both Hoste [15] and Hartley [10] discuss the relationship between a_0 and the complete graph which has a vertex for each component of $\tilde{L}$. (This relationship is simply the application of the matrix-tree theorem (see [6]) to the matrix λ.) The fact that $a_0 \neq 0$ implies that there is a spanning tree in this complete graph with the property that whenever two vertices are connected by an edge in the tree, the linking number of the corresponding components of $\tilde{L}$ is nonzero.

The desired listing of the components of $\tilde{L} - L$ can be obtained by listing first those components that correspond to extremal vertices of this spanning tree (in any order), next those components that correspond to vertices that are extremal among the remaining vertices (in any order), and so on. Q.E.D.

PROPOSITION (3.4). *Suppose i is an extendable I-equivalence between links L_0 and L_1. Then the $\pm$ in Theorem 3 is unnecessary for i.*

PROOF. It suffices to prove this in case i is strongly extendable. Suppose $\tilde{i}$ extends i. That the $\pm$ in Theorem 3 is unnecessary for $\tilde{i}$ itself follows immediately from the fact that linking numbers are I-equivalence invariants, since this implies that $\tilde{L}_0$ and $\tilde{L}_1$ have the same nonzero a_0. Lemma (3.3) and Torres' second relation imply that the relationship between ∇_{L_0} and ∇_{L_1} is determined by the relationship between $\nabla_{\tilde{L}_0}$ and $\nabla_{\tilde{L}_1}$. Q.E.D.

Theorem 2 follows from (1.3) and (1.4), the corresponding result for the Alexander polynomial [9, 17, 31], and

PROPOSITION (3.5). *Piecewise linear I-equivalences are extendable.*

PROOF. It suffices to show this for isotopies and concordances, since these generate the relation of piecewise linear I-equivalence [36].

For isotopies, it suffices to consider two links L_0 and L_1, L_1 obtained from L_0 by connected-summing a knot into one of the components of L_0. We may suppose that this knot is contained in a sphere of arbitrarily small radius, centered at some point of L_0. Let x be as in Lemma (2.1). Clearly then a component K may be adjoined to L_0 in such a way that its linking number with every component of L_0 is x, and K does not meet this small sphere. The isotopy between L_0 and L_1 then extends to an isotopy between $L_0 \cup K$ and $L_1 \cup K$.

For concordances, it suffices to consider two links L_0 and L_1, L_1 the "band sum" of L_0 with a trivial link T [45]. Up to ambient isotopy, the (finitely many) bands involved in this band sum may be assumed to be of arbitrarily small widths; also, T may be assumed to be situated at some distance from L_0. There is, then, a regular projection of L_1 in the plane with this property: on each component of L_0 there is some arc whose image in the projection does not contain any points that lie within the image of any band, or within the image of any component of T. Let x be as in Lemma (2.1), and let K be a knot which is contained in some horizontal plane far above all of L_1, except that over each of these arcs on the components of L_0 a portion of K drops straight down to encircle that component x times. Clearly then the concordance between L_0 and L_1 extends to a concordance between $L_0 \cup K$ and $L_1 \cup K$. Q.E.D.

Another interesting equivalence relation between links is *F*-isotopy. Suppose a link L_1 can be obtained from a link L_0 by choosing a regular neighborhood of some component of L_0 (the neighborhood must not meet any other components of L_0), and replacing that component by some knot which is homologous to it in that regular neighborhood. Then L_0 and L_1 are related by an *elementary F-isotopy*. The general relation of F-isotopy is simply the equivalence relation generated by the elementary F-isotopies. F-isotopy was introduced by Smythe [12, 38], who studied the relationship between the Alexander modules of F-isotopic links, and also investigated by Giffen (unpublished), who proved that F-isotopic links are topologically I-equivalent.

PROPOSITION (3.6). *If L_0 and L_1 are F-isotopic then there is an extendable I-equivalence between them.*

PROOF. It suffices to prove this in case L_1 is obtained from L_0 through an elementary F-isotopy. In this case they differ only inside a certain solid torus, and they also have the property that if K is any knot outside this solid torus, each component of L_1 has the same linking number with K as the corresponding component of L_0. If x is as in Lemma (2.1), then such a knot K may be adjoined to L_0 and L_1 in such a way that its linking number with every component of L_0

and L_1 is x. Then $L_0 \cup K$ and $L_1 \cup K$ will be F-isotopic, and hence I-equivalent (by Giffen's theorem), so there is an extendable I-equivalence between L_0 and L_1. Q.E.D.

4. Simplifying the potential function

Let L be a link in S^3. Suppose we choose a particular Alexander polynomial of L, and factor it as

$$\Delta_L = u p_1^{s_1} \cdots p_q^{s_q},$$

where u is a unit, $p_1, \ldots, p_q$ are pairwise non-associate irreducibles of $Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$, and $s_1, \ldots, s_q$ are positive integers. Replacing u by $-u$ if necessary, we may assume that $\epsilon(p_i) \geq 0 \; \forall i$.

Note that since Δ_L and $\bar{\Delta}_L$ are associates, for each i $\bar{p}_i^{s_i}$ also divides Δ_L. Hence for each i one of these possibilities must occur: either there is a $j \neq i$ with $\bar{p}_i$ an associate of p_j and $s_i = s_j$, or else $\bar{p}_i$ is an associate of p_i. Replacing u by another unit if necessary, we may assume that whenever $i \neq j$, $\epsilon(p_i) = 1$, and p_j is an associate of $\bar{p}_i$, p_j is actually equal to $\bar{p}_i$. By Lemma (3.2), we may similarly assume that whenever $\epsilon(p_i) = 1$ and p_i is an associate of $\bar{p}_i$, p_i actually equals $\bar{p}_i$.

We may now factor Δ_L as

$$\Delta_L = P_1 P_2 \bar{P}_2 P_3 \bar{P}_3 P_4 P_5$$

in the following way. P_1 is the product of those p_i such that s_i is odd, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i involves only one of the variables $t_1, \ldots, t_\mu$. P_2 is the product of three types of factors: those $p_i^{(s_i-1)/2}$ such that s_i is odd, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i involves only one variable; those $p_i^{s_i/2}$ such that s_i is even, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i only involves one variable; and one of each pair $p_i^{s_i}$, $p_j^{s_j}$ such that $i \neq j$, $p_j = \bar{p}_i$, $\epsilon(p_i) = 1$, and some associate of p_i only involves one variable. P_4 is the product of those p_i such that s_i is odd, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i involves more than one variable. P_3 is the product of three types of factors: those $p_i^{(s_i-1)/2}$ such that s_i is odd, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i involves more than one variable; those $p_i^{s_i/2}$ such that s_i is even, $\epsilon(p_i) = 1$, $p_i = \bar{p}_i$, and p_i involves more than one variable; and one of each pair $p_i^{s_i}$, $p_j^{s_j}$ such that $i \neq j$, $p_j = \bar{p}_i$, $\epsilon(p_i) = 1$, and no associate of p_i involves only one variable. By Lemma (3.2), the p_i with $\epsilon(p_i) = 1$ and $p_i = \bar{p}_i$ are uniquely determined by these two properties. Furthermore, if $i \neq j$ and $p_j = \bar{p}_i$ then the product $p_i p_j$ is obviously unchanged if p_i is replaced by an associate and p_j is replaced by the new $\bar{p}_i$. Thus $P_1, P_2\bar{P}_2, P_3\bar{P}_3$, and P_4 are uniquely determined, despite the fact that Δ_L itself is not. That is, if Δ_L is replaced by some other Alexander polynomial of L, then so long as the irreducible factors with $\epsilon(p_i) = 1$ are chosen so that $p_j = \bar{p}_i$ whenever p_j is an associate of $\bar{p}_i$, the corresponding factorization $P_1(P_2\bar{P}_2)(P_3\bar{P}_3)P_4P_5$ will differ from that of Δ_L only in P_5.

We define quotients ∇_L^{iso}, ∇_L^{conc}, ∇_L^{pli}, and ∇_L^{topi} of the potential function as follows.

$$\nabla_L^{iso} = \nabla_L/P_1(t_1^2, \ldots, t_\mu^2)P_2(t_1^2, \ldots, t_\mu^2)\bar{P}_2(t_1^2, \ldots, t_\mu^2),$$

$$\nabla_L^{conc} = \nabla_L/P_2(t_1^2, \ldots, t_\mu^2)\bar{P}_2(t_1^2, \ldots, t_\mu^2)P_3(t_1^2, \ldots, t_\mu^2)\bar{P}_3(t_1^2, \ldots, t_\mu^2),$$

$$\nabla_L^{pli} = \nabla_L^{conc}/P_1(t_1^2, \ldots, t_\mu^2), \text{ and}$$

$$\nabla_L^{topi} = \nabla_L^{pli}/P_4(t_1^2, \ldots, t_\mu^2).$$

Then Theorem 1 of the introduction implies that isotopic links have the same ∇^{iso}, Theorem 2 implies that concordant links have the same ∇^{conc}, Theorems 1 and 2 imply that p.l. I-equivalent links have the same ∇^{pli}, and Theorem 3 implies that topologically I-equivalent links have the same ∇^{topi} (at least up to sign).

5. Sublinks

The astute reader may have noticed that in defining ∇_L^{iso} in §4 we did not take advantage of the full strength of Theorem 1: instead of dividing ∇_L by $P_1(t_1^2, \ldots, t_\mu^2)P_2(t_1^2, \ldots, t_\mu^2)\bar{P}_2(t_1^2, \ldots, t_\mu^2)$ we could simply have considered the element $\nabla_L/\Pi\nabla_{K_i}$ of the field of quotients of $Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ as an invariant of the isotopy class of L. The reason we did not do this is that it turns out that there are analogous weaknesses in the definitions of ∇_L^{conc}, ∇_L^{pli}, and ∇_L^{topi} (at least as far as extendable I-equivalences are concerned) which are, unfortunately, not so easily corrected by a simple change of definition. The weaknesses we are referring to amount to the observation that there may be useful information contained in the relationship between the potential functions of a link and its sublinks, which is not contained in any single one of these potential functions.

For $1 \leq i_1 < \cdots < i_\nu \leq \mu$ let $\phi_{i_1, \ldots, i_\nu}$: $Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}] \to Z[t_{i_1}, \ldots, t_{i_\nu}, t_{i_1}^{-1}, \ldots, t_{i_\nu}^{-1}]$ be given by $\phi_{i_1, \ldots, i_\nu}(t_{i_j}) = t_{i_j}$ for $1 \leq j \leq \nu$, and $\phi_{i_1, \ldots, i_\nu}(t_i) = 1$ for $i \notin \{i_1, \ldots, i_\nu\}$.

THEOREM (5.1). *Suppose there is an extendable I-equivalence between links L_0 and L_1 with nonzero potential functions. By Theorem 3 and Proposition* (3.4), *there are then $F_0, F_1 \in Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ with $\epsilon(F_0) = 1 = \epsilon(F_1)$, $F_0 = \bar{F}_0$, $F_1 = \bar{F}_1$, and*

$$\nabla_{L_0} \cdot F_0(t_1^2, \ldots, t_\mu^2) = \nabla_{L_1} \cdot F_1(t_1^2, \ldots, t_\mu^2).$$

(Note that this determines F_0 and F_1, except for common factors.) Then for

every choice of $1 \leq i_1 < \cdots < i_\nu \leq \mu$,

$$\nabla_{\bar{L}_0} \cdot \phi_{i_1,\ldots,i_\nu}(F_0(t_1^2, \ldots, t_\mu^2)) = \nabla_{\bar{L}_1} \cdot \phi_{i_1,\ldots,i_\nu}(F_1(t_1^2, \ldots, t_\mu^2)),$$

where $\bar{L}_0 = K_{i_1 0} \cup \cdots \cup K_{i_\nu 0}$ *and* $\bar{L}_1 = K_{i_1 1} \cup \cdots \cup K_{i_\nu 1}$.

PROOF. Recall that linking numbers are invariants of I-equivalence.

Suppose that L_0 itself has $a_0 \neq 0$; then so does L_1. We may apply Lemma (3.3) (with L_0, L_1 playing the role of $\tilde{L}$ and $\bar{L}_0$, $\bar{L}_1$ playing the role of L) and use Torres' second relation repeatedly, cancelling like factors after each application, to conclude that the relationship between $\nabla_{\bar{L}_0}$ and $\nabla_{\bar{L}_1}$ is determined by the relationship between ∇_{L_0} and ∇_{L_1}.

More generally, suppose the I-equivalence between L_0 and L_1 extends to an I-equivalence between links $\tilde{L}_0$ and $\tilde{L}_1$ which have $a_0 \neq 0$. The result of the theorem follows by applying the case already discussed twice: once to L_0, L_1 as sublinks of $\tilde{L}_0$, $\tilde{L}_1$; and again to $\bar{L}_0$, $\bar{L}_1$ as sublinks of $\tilde{L}_0$, $\tilde{L}_1$.

The general case follows, by applying this last case repeatedly. Q.E.D.

When L_0 has $a_0 \neq 0$, Theorem (5.1) follows directly from Torres' second relation and so represents no new information. However, when L_0 has $a_0 = 0$, Theorem (5.1) can be valuable; for instance, there are several pairs of two-component links of linking number zero listed by Conway [5] which have the same potential function, but can be distinguished by Theorem (5.1) (e.g., 212 and 21,2,2-).

6. The reduced potential function

It is obvious that analogues of Theorems 1, 2, and 3 hold for the reduced potential function; they are deduced from the Theorems simply by applying the homomorphism π. These lead to simplified versions D_L^{iso}, D_L^{conc}, D_L^{pli}, and D_L^{topi} of D_L, analogous to the simplifications ∇_L^{iso}, ∇_L^{conc}, ∇_L^{pli}, and ∇_L^{topi} of §4; we leave the details of their definitions to the reader.

The reduced potential function of a μ-component link L may be written as a polynomial

$$D_L = z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n})$$

in $z = t - t^{-1}$ [14, 16, 27]; if $\mu = 1$ then a_0 is 1. A partial converse of this follows from realization theorems of Hosokawa [13] and Seifert [37] for the Alexander polynomial.

PROPOSITION (6.1). *Let* D *be a polynomial* $D = z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n})$, *with* $a_0 = \pm 1$ *if* $\mu = 1$. *Then at least one of* D *and* $-D$ *is the reduced potential function of some* μ*-component link.*

PROOF. Let

$$f(t) = a_0 + a_2 t^{-1}(t-1)^2 + a_4 t^{-2}(t-1)^4 + \cdots + a_{2n} t^{-n}(t-1)^{2n},$$

so that $D = t^{1-\mu}(t^2-1)^{\mu-1} f(t^2)$. Note that $f = \bar{f}$, and the degree of f is $2n$.

According to Seifert [37], if $\mu = 1$ then there is some knot K with f as Alexander polynomial. Then since $D = \bar{D}$, it follows from (1.4) that $D_K = \pm D$.

On the other hand if $\mu \geq 2$ then according to Hosokawa [13] there is a μ-component link L with Alexander polynomial Δ_L such that $\pi(\Delta_L) = (t-1)^{\mu-2} \cdot f(t)$. Since $D = (-1)^{\mu-1}\bar{D}$, it follows from (1.3) that $D_L = \pm D$. Q.E.D.

PROPOSITION (6.2). *The simplifications D_L^{iso}, D_L^{conc}, D_L^{pli}, and D_L^{topi} can also be expressed in the form $z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n})$. We write*

$$D_L^{iso} = z^{\mu-1} \cdot (a_0^{iso} + a_2^{iso} z^2 + \cdots + a_{2n}^{iso} z^{2n}),$$

and so on. Furthermore, $a_0 = a_0^{iso} = a_0^{conc} = a_0^{pli} = a_0^{topi}$, and for $i > 0$ $a_{2i} \equiv a_{2i}^{iso} \equiv a_{2i}^{pli} \equiv a_{2i}^{topi}$ (mod $(a_0, \ldots, a_{2i-2})$) while $a_{2i} \equiv a_{2i}^{conc}$ (mod $(a_0, \ldots, a_{2i-4}, 2a_{2i-2})$).

PROOF. Observe that each of the simplified versions of D_L is a quotient $D_L/P(t^2)$ for some $P \in Z[t, t^{-1}]$ with $\epsilon(P) = 1$ and $P = \bar{P}$. According to Seifert [37], such a P is the Alexander polynomial of some knot, so by (1.4) $P(t^2)$ is the reduced potential function of some knot; hence $P(t^2) = 1 + b_2 z^2 + \cdots + b_{2m} z^{2m}$ for some $m \leq n$. It follows immediately that each of D_L^{iso}, D_L^{conc}, D_L^{pli}, D_L^{topi} can be expressed in the form $z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n})$, with each coefficient a_{2i} congruent to the corresponding coefficient in D_L, modulo the coefficients of preceding terms.

In the special case of D_L^{conc}, $P = Q\bar{Q}$ for some Laurent polynomial Q. According to Nakagawa [31], such a P is the Alexander polynomial of a slice knot, so by (1.4) $P(t^2) = 1 + b_2 z^2 + \cdots + b_{2m} z^{2m}$ is the reduced potential function of some slice knot. Such a knot has Arf invariant zero, and according to Kauffman [16] this implies that b_2 is even. Q.E.D.

The last sentence of this proposition immediately implies that some of the information in the simplified versions of D_L can be obtained from D_L without troubling to factor it.

COROLLARY (6.3). *Suppose L_0 and L_1 are μ-component links with $D_{L_j} = z^{\mu-1} \cdot (a_0(L_j) + a_2(L_j) z^2 + \cdots)$ for $j \in \{0, 1\}$. Then:*

(a) *if L_0 and L_1 are concordant, $a_{2i}(L_0) \equiv a_{2i}(L_1)$ (modulo $(a_0(L_0), \ldots, a_{2i-4}(L_0), 2a_{2i-2}(L_0))$) $\forall i$;*

(b) *if* L_0 *and* L_1 *are p.l. I-equivalent,* $a_{2i}(L_0) \equiv a_{2i}(L_1)$ (*modulo* $(a_0(L_0), \ldots, a_{2i-2}(L_0))) \forall i$; *and*
(c) *if* L_0 *and* L_1 *are topologically I-equivalent,* $a_{2i}(L_0) \equiv \pm a_{2i}(L_1)$ (*modulo* $(a_0(L_0), \ldots, a_{2i-2}(L_0))) \forall i$.

Part (b) was stated by Cochran [3].

A useful invariant of link type related to the reduced potential function is the *determinant* $D_L(\sqrt{-1})$ (classically, the determinant was defined using the Alexander polynomial rather than the potential function). Obviously analogous invariants can be obtained from the simplified potential functions: $D_L^{iso}(\sqrt{-1})$, $D_L^{conc}(\sqrt{-1})$, and $D_L^{pli}(\sqrt{-1})$ will be invariants of the appropriate equivalence relations, and $D_L^{topi}(\sqrt{-1})$ will be an invariant of topological I-equivalence up to sign. In analogy with Corollary (6.3) is the fact that $D_L^{conc}(\sqrt{-1})$ will differ from $D_L(\sqrt{-1})$ only by odd square factors, and the other three will differ from $D_L(\sqrt{-1})$ only by odd factors. An example of the use of these results may be found in [44], where we exhibited an infinite set of two-component links whose I-equivalence classes may be distinguished by their determinants, even though the links have the same Murasugi signatures, Murasugi nullities, and Milnor $\bar{\mu}$-invariants.

7. Expanding the potential function

The expression of the reduced potential function in the form $D_L = z^{\mu-1} \cdot (a_0 + a_2 z^2 + \cdots)$ is pleasant and useful, as it serves to highlight several properties of D_L: a_0 is determined by the linking numbers in L (and more generally, all the a_i are related to Milnor's $\bar{\mu}$-invariants; see §8), a_2 is related to the Arf invariant (we'll discuss this in more detail in §9), and some of the concordance- or I-equivalence-invariant information in D_L is easily gleaned from the a_i (see Corollary (6.3)). The initial impetus for this paper was our hope of finding an expression of the potential function ∇_L that would be analogous to this expression of D_L as a polynomial in z and would be equally useful and revealing.

Unfortunately, the obvious possibility — that ∇_L might be expressible as a polynomial in $t_1 - t_1^{-1}, \ldots, t_\mu - t_\mu^{-1}$ — turns out to be simply untrue; examples of this can easily be found in the tables of [5]. However, a slightly more complicated expression can be found, suggested by Smythe's [38] and Massey's [24] work with the Taylor series expansion of the Alexander polynomial.

Given a link $L = K_1 \cup \cdots \cup K_\mu$ of $\mu \geq 2$ components, define integers $q_1, \ldots, q_\mu$ by $q_i = -1 + \ell(K_i, L - K_i)$. Observing that for any Alexander polynomial of L these $q_1, \ldots, q_\mu$ will be congruent (mod 2) to the $n_1, \ldots, n_\mu$ of (1.1) and (1.3), we conclude that $t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L$ is a Laurent polynomial which involves only even powers of $t_1, \ldots, t_\mu$. Hence it may be expressed as a power series

$$t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L = \Sigma\, c(r_1, \ldots, r_\mu)(t_1^2 - 1)^{r_1} \cdots (t_\mu^2 - 1)^{r_\mu}$$

in the usual way, i.e., by replacing t_i^2 by $1 + (t_i^2 - 1)$ and t_i^{-2} by $1 - (t_i^2 - 1) + (t_i^2 - 1)^2 - + \cdots = \Sigma(-1)^n (t_i^2 - 1)^n$ and multiplying out.

We proceed to discuss properties of this expression of ∇_L that are analogous to the results mentioned in §6. We know of no analogue of Proposition (6.1); as far as we know there is no general characterization of the multivariate Laurent polynomials that arise as the Alexander polynomials of links (see [11]).

To state an analogue of Proposition (6.2) we need to establish some notation. If $r_1, \ldots, r_\mu \geq 0$ let $d^*(r_1, \ldots, r_\mu) \geq 0$ be the g.c.d. of those $c(s_1, \ldots, s_\mu)$ with $s_i \leq r_i\ \forall i$ and $\Sigma\, s_i \leq -2 + \Sigma\, r_i$, and let $\hat{d}(r_1, \ldots, r_\mu)$ be the greatest common divisor of $\{c(s_1, \ldots, s_\mu) \mid s_i \leq r_i\ \forall i \text{ and } \Sigma\, s_i < -2 + \Sigma\, r_i\} \cup \{2c(s_1, \ldots, s_\mu) \mid s_i \leq r_i\ \forall i \text{ and } \Sigma\, s_i = -2 + \Sigma\, r_i\}$.

PROPOSITION (7.1). *The multiples* $t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L^{iso}$, *etc. of the simplified versions of the potential function can all be expressed as power series in* $t_1^2 - 1, \ldots, t_\mu^2 - 1$. *We write*

$$t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L^{iso} = \Sigma\, c_L^{iso}(r_1, \ldots, r_\mu)(t_1^2 - 1)^{r_1} \cdots (t_\mu^2 - 1)^{r_\mu},$$

etc., suppressing the subindex L *when this will cause no confusion. Furthermore, when* $r_1, \ldots, r_\mu \geq 0$, $c(r_1, \ldots, r_\mu) \equiv c^{iso}(r_1, \ldots, r_\mu) \equiv c^{pli}(r_1, \ldots, r_\mu) \equiv c^{topi}(r_1, \ldots, r_\mu)$ *(mod* $d^*(r_1, \ldots, r_\mu)$*) and* $c(r_1, \ldots, r_\mu) \equiv c^{conc}(r_1, \ldots, r_\mu)$ *(mod* $\hat{d}(r_1, \ldots, r_\mu)$*).*

PROOF. The first assertion follows immediately from the fact that $t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L^{iso}$, etc. are all Laurent polynomials in $t_1^2, \ldots, t_\mu^2$.

Now recall that each of ∇_L^{iso}, ∇_L^{conc}, ∇_L^{pli}, and ∇_L^{topi} is a quotient $\nabla_L / P(t_1^2, \ldots, t_\mu^2)$, where $P \in Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ has $\epsilon(P) = 1$ and $P = \bar{P}$. Suppose we express $P(t_1^2, \ldots, t_\mu^2)$ as a power series

$$P(t_1^2, \ldots, t_\mu^2) = b_0 + \Sigma_i\, b_i (t_i^2 - 1) + \cdots.$$

Since $\epsilon(P) = 1$, $b_0 = 1$. Furthermore,

$$\begin{aligned} \bar{P}(t_1^2, \ldots, t_\mu^2) &= P(t_1^{-2}, \ldots, t_\mu^{-2}) \\ &= 1 + \Sigma_i\, b_i (t_i^{-2} - 1) + \cdots \\ &= 1 - \Sigma_i\, b_i (t_i^2 - 1) + \cdots, \end{aligned}$$

so since $P = \bar{P}$, necessarily $b_1, \ldots, b_\mu$ are all 0.

In the special case of ∇_L^{conc}, recall that P is of the form $P = Q\bar{Q}$ for some $Q \in Z[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$. Suppose we write

$$Q(t_1^2, \ldots, t_\mu^2) = 1 + \Sigma_i\, e_i (t_i^2 - 1) + \Sigma_{i \leq j}\, e_{ij} (t_i^2 - 1)(t_j^2 - 1) + \cdots.$$

Then

$$\begin{aligned}\bar{Q}(t_1^2,\ldots,t_\mu^2) &= 1+\Sigma_i e_i(t_i^{-2}-1)+\Sigma_{i\le j} e_{ij}(t_i^{-2}-1)(t_j^{-2}-1)+\cdots\\ &= 1-\Sigma_i e_i(t_i^2-1)+\Sigma_i e_i(t_i^2-1)^2\\ &\qquad +\Sigma_{i\le j} e_{ij}(t_i^2-1)(t_j^2-1)+\cdots,\end{aligned}$$

so

$$\begin{aligned}P(t_1^2,\ldots,t_\mu^2) &= Q(t_1^2,\ldots,t_\mu^2)\bar{Q}(t_1^2,\ldots,t_\mu^2)\\ &= 1+\Sigma_i(e_i-e_i^2)(t_i^2-1)^2+\Sigma_{i\le j}2e_{ij}(t_i^2-1)(t_j^2-1)\\ &\qquad -\Sigma_{i<j}2e_ie_j(t_i^2-1)(t_j^2-1)+\cdots,\end{aligned}$$

in which all the coefficients of terms of degree two are even. Q.E.D.

The integers $q_1, \ldots, q_\mu$ will coincide for I-equivalent links, as linking numbers are I-equivalence invariants. It follows that the various coefficients $c^{iso}(r_1, \ldots, r_\mu)$, $c^{conc}(r_1, \ldots, r_\mu)$, and $c^{pli}(r_1, \ldots, r_\mu)$ are invariants for the appropriate equivalence relations, and the $c^{topi}(r_1, \ldots, r_\mu)$ are (at least up to a single sign change) invariants for topological I-equivalence. In analogy with Corollary (6.3), it follows from Proposition (7.1) that some of the information in these invariants may be obtained from ∇_L without factoring it.

COROLLARY (7.2). *Suppose L_0 and L_1 are μ-component links. Then:*
(a) *if L_0 and L_1 are concordant, $c_{L_0}(r_1, \ldots, r_\mu) \equiv c_{L_1}(r_1, \ldots, r_\mu)$ (mod $\hat{d}(r_1, \ldots, r_\mu)$) $\forall\, r_1, \ldots, r_\mu \geq 0$;*
(b) *if L_0 and L_1 are p.l. I-equivalent, $c_{L_0}(r_1, \ldots, r_\mu) \equiv c_{L_1}(r_1, \ldots, r_\mu)$ (mod $d^*(r_1, \ldots, r_\mu)$) $\forall\, r_1, \ldots, r_\mu \geq 0$; and*
(c) *if L_0 and L_1 are topologically I-equivalent, $c_{L_0}(r_1, \ldots, r_\mu) \equiv \pm c_{L_1}(r_1, \ldots, r_\mu)$ (mod $d^*(r_1, \ldots, r_\mu)$) $\forall\, r_1, \ldots, r_\mu \geq 0$.*

The relationship between the coefficients $c(r_1, \ldots, r_\mu)$ of this section and the coefficients $a_0, a_1, \ldots$ of §6 is described in the following proposition (we adopt the convention that $a_k = 0$ for odd k).

PROPOSITION (7.3). *Suppose L is a μ-component link. For $r \geq 0$ let c_r be the sum of all the $c(r_1, \ldots, r_\mu)$ with $\Sigma r_i = r$. Then $c_r = 0$ for $r < \mu - 2$, and $c_r \equiv a_{r-\mu+2}$ (modulo $(a_0, \ldots, a_{r-\mu+3})$) for $r \geq \mu - 2$.*

PROOF. Let $q = \Sigma q_i$; note that $q \equiv \mu \pmod 2$. Hence

$$t^{\mu-q-2} \cdot \Sigma c_r(t^2-1)^{r+1}$$

may be expanded as a power series in $t^2 - 1$; clearly if this is done then for each r

the coefficient of $(t^2-1)^{r+1}$ will be congruent to c_r (modulo $(c_0, \ldots, c_{r-1})$). On the other hand,

$$\begin{aligned}
&t^{\mu-q-2} \cdot \Sigma\, c_r (t^2-1)^{r+1} \\
&= t^{-1}(t^2-1) \cdot \pi(t_1^{q_1} \cdots t_\mu^{q_\mu} \nabla_L) \cdot t^{\mu-1-q} \\
&= (t-t^{-1})\, \pi(\nabla_L) t^{\mu-1} \\
&= t^{\mu-1} z^{\mu-1} \cdot (a_0 + a_2 z^2 + a_4 z^4 + \cdots) \\
&= (t^2-1)^{\mu-1} \cdot (a_0 + a_2 t^{-2}(t^2-1)^2 + a_4 t^{-4}(t^2-1)^4 + \cdots),
\end{aligned}$$

and clearly if this is expanded as a power series in t^2-1 then for each r the coefficient of $(t^2-1)^{r+1}$ will be congruent to $a_{r-\mu+2}$, modulo the preceding coefficients. Q.E.D.

8. Milnor's $\bar{\mu}$-invariants

In this section we will restrict our attention to links of two or more components.

A *regular projection* of a link $L = K_1 \cup \cdots \cup K_\mu$ in the plane is a projection with only finitely many singularities, all of which are transverse double points ("crossings"). These crossings break the image of L into a collection of pairwise disjoint, simple arcs; for each i, the arcs constituting the image of K_i may be indexed in a manner consistent with the orientation of K_i. (Technically, we regard the specification of such an indexing as part of the specification of a regular projection.)

Given a regular projection of L, Milnor [26] associated an integer $\mu(i_1, \ldots, i_p)$ to each sequence $i_1, \ldots, i_p$ of $p \geq 2$ elements of $\{1, \ldots, \mu\}$. (We hope that the two uses of the letter "μ" will cause no confusion.) We will not repeat the definition of these integers here, as it is rather complicated.

Three basic properties of the integers $\mu(i_1, \ldots, i_p)$ are that $\mu(i,j) = 0$ whenever $i=j$, $\mu(i,j) = \ell(K_i, K_j)$ whenever $i \neq j$, and

$$\mu(i, \ldots, i, j) = \binom{\mu(i,j)}{p-1}$$

whenever $(i, \ldots, i, j)$ consists of $p-1$ is and one j (cf. [26] and [7]). The integers $\mu(i_1, \ldots, i_p)$, $p \geq 3$, are not invariants of the link type of L, in general; rather, they depend on the choice of the particular projection of L used in their definition. Milnor [26] showed, though, that if for each sequence $(i_1, \ldots, i_p)$ an integer $\Delta(i_1, \ldots, i_p) \geq 0$ is defined to be the greatest common divisor of those $\mu(j_1, \ldots, j_q)$ such that $(j_1, \ldots, j_q)$ is a cyclic permutation of some proper subsequence of $(i_1, \ldots, i_p)$, then $\Delta(i_1, \ldots, i_p)$ and the congruence class $\bar{\mu}(i_1, \ldots, i_p)$ of $\mu(i_1, \ldots, i_p)$ modulo $\Delta(i_1, \ldots, i_p)$ are invariants of L not only under ambient isotopy, but also under the coarser equivalence relation of (topological) isotopy. It follows from a result of Casson [2] that they are, in fact, invariant under the still coarser relation of topological I-equivalence. More recent results (see [3, 4, 21, 22, 40, 44] and below) indicate that the integers $\mu(i_1, \ldots, i_p)$ contain information about the I-equivalence class of a link that is not detected by the congruence classes $\bar{\mu}(i_1, \ldots, i_p)$, and that at least in special cases more sensitive versions of the $\mu(i_1, \ldots, i_p)$ can be defined.

Another useful property of these integers is the following direct consequence of Fox's "chain rule" [7].

LEMMA (8.1). *Let $L = K_1 \cup \cdots \cup K_\mu$ consist of the first μ components of a link $\tilde{L}$. For $i_1, \ldots, i_p \in \{1, \ldots, \mu\}$ let $\tilde{\mu}(i_1, \ldots, i_p)$ be the integer associated to a particular regular projection of $\tilde{L}$, and $\mu(i_1, \ldots, i_p)$ the integer associated to the regular projection of L obtained in the obvious way from that of $\tilde{L}$. Then $\mu(i_1, \ldots, i_p) = \tilde{\mu}(i_1, \ldots, i_p)$.*

Several authors [12, 30, 38, 43] have observed that the relationship between the Alexander polynomial and linking numbers extends to a relationship between Δ and the various integers $\mu(i_1, \ldots, i_p)$. In the next theorem we state the corresponding relationship between the potential function and these integers.

Let $\mathcal{S} = (\mathcal{S}_{ij})$ be the $\mu \times \mu$ matrix with diagonal entries given by

$$\mathcal{S}_{ii} = \sum_{p=1}^{\infty} \Sigma\, \mu(i_1, \ldots, i_p, i) \cdot \Pi(t_{i_k}^2 - 1),$$

the sum Σ taken over the set of p-tuples $(i_1, \ldots, i_p)$ with $i_p \neq i$, and other entries given by

$$\mathcal{S}_{ij} = -\mu(j, i) \cdot (t_i^2 - 1) - \sum_{p=1}^{\infty} \Sigma\, \mu(i_1, \ldots, i_p, j, i) \cdot (t_i^2 - 1) \cdot \Pi(t_{i_k}^2 - 1),$$

the sum Σ now taken over the set of all p-tuples $(i_1, \ldots, i_p)$ of elements of $\{1, \ldots, \mu\}$. Then

THEOREM (8.2). *For every choice of $i, j \in \{1, \ldots, \mu\}$ there is a power series u_{ij} in $t_1^2 - 1, \ldots, t_\mu^2 - 1$, with constant term $(-1)^{i+j}$, such that*

$$t_1^{q_1} \cdots t_\mu^{q_\mu} \circ \nabla_L = u_{ij} \cdot \det \mathcal{S}^{(ij)} / (t_j^2 - 1).$$

PROOF. Let $\mathcal{M}$ be the matrix obtained from $-\mathcal{S}$ by replacing each t_r^2 in an entry of $-\mathcal{S}$ by t_r. It follows from [43, Theorem (4.1)] that for every choice of $i, j \in \{1, \ldots, \mu\}$, Δ_L is a multiple of $\det \mathcal{M}^{(ij)} / (t_j - 1)$ by some unit of the power series ring $Z[[t_1 - 1, \ldots, t_\mu - 1]]$ (i.e., some power series in $t_1 - 1, \ldots, t_\mu - 1$ with constant term ± 1). With (1.3), this implies that Theorem (8.2) holds up to sign. To complete the proof, then, we need only show that the u_{ij} can be chosen so that $\epsilon(u_{ij}) = (-1)^{i+j}$. Of course, if $\nabla_L = 0$ we may simply replace each u_{ij} by $(-1)^{i+j}$, if necessary; so we proceed under the assumption that $\nabla_L \neq 0$.

If the matrix λ associated to L has det $\lambda^{(11)} \neq 0$, we may use this to compute $\epsilon(u_{ij})$:

$$\begin{aligned}
&(-1)^{i+j}\, det\, \lambda^{(ij)} \\
&= a_0 \\
&= \epsilon\left[\frac{\pi\nabla_L}{(t-t^{-1})^{\mu-2}}\right] \\
&= \epsilon(u_{ij})\cdot\epsilon\left[\frac{\pi(det\, \mathcal{S}^{(ij)})}{(t-t^{-1})^{\mu-1}}\right] \\
&= \epsilon(u_{ij})\cdot\epsilon\left[det\,(\pi\mathcal{S}^{(ij)}/(t-t^{-1}))\right] \\
&= \epsilon(u_{ij})\cdot det\, \lambda^{(ij)}.
\end{aligned}$$

On the other hand, suppose L has det $\lambda^{(11)} = 0$. Let $\widetilde{L} = L \cup K_{\mu+1}$ be a link of the kind mentioned in Lemma (2.1), with $\ell(K_i, K_{\mu+1}) = x \quad \forall i \leq \mu$ and det $\widetilde{\lambda}^{(11)} \neq 0$. Note that then $x \neq 0$, and the integers $\widetilde{q}_1, \ldots, \widetilde{q}_{\mu+1}$ associated to $\widetilde{L}$ are related to the integers $q_1, \ldots, q_\mu$ associated to L by $\widetilde{q}_i = q_i + x\ \forall i \leq \mu$. As we just saw, there are units $\widetilde{u}_{ij} \in Z[[t_1-1, \ldots, t_{\mu+1}-1]]$ with

$$t_1^{\widetilde{q}_1}\cdots t_{\mu+1}^{\widetilde{q}_{\mu+1}}\cdot\nabla_{\widetilde{L}} = \widetilde{u}_{ij}\cdot det\, \widetilde{\mathcal{S}}^{(ij)}/(t_j^2-1)$$

and $\epsilon(\widetilde{u}_{ij}) = (-1)^{i+j}$.

Let $\phi\colon\ Z[[t_1-1, \ldots, t_{\mu+1}-1]] \to Z[[t_1-1, \ldots, t_\mu-1]]$ be the homomorphism given by $\phi(t_i-1) = t_i - 1$ for $i \leq \mu$, and $\phi(t_{\mu+1}-1) = 0$. By Lemma (8.1), $\phi(\widetilde{\mathcal{S}}_{ij}) = \mathcal{S}_{ij}$ for $i, j \in \{1, \ldots, \mu\}$; also, clearly $\phi(\widetilde{\mathcal{S}}_{\mu+1\, j}) = 0$ for $j \in \{1, \ldots, \mu\}$. Thus for $i, j \in \{1, \ldots, \mu\}$,

$$\begin{aligned}
&((\prod_{i=1}^{\mu} t_i^{2x}) - 1)\cdot u_{ij}\cdot det\, \mathcal{S}^{(ij)}/(t_j^2-1) \\
&= ((\prod_{i=1}^{\mu} t_i^{x}) - (\prod_{i=1}^{\mu} t_i^{-x}))\cdot t_1^{\widetilde{q}_1}\cdots t_\mu^{\widetilde{q}_\mu}\cdot\nabla_L \\
&= \phi(t_1^{\widetilde{q}_1}\cdots t_{\mu+1}^{\widetilde{q}_{\mu+1}}\cdot\nabla_{\widetilde{L}}) \\
&= \phi(\widetilde{u}_{ij})\cdot\phi(det\, \widetilde{\mathcal{S}}^{(ij)}/(t_j^2-1)) \\
&= \phi(\widetilde{u}_{ij})\cdot\phi(\widetilde{\mathcal{S}}_{\mu+1\,\mu+1})\cdot det\, \mathcal{S}^{(ij)}/(t_j^2-1),
\end{aligned}$$

and so

$$\begin{aligned}
&((\prod_{i=1}^{\mu} t_i^{2x}) - 1)\cdot u_{ij} \\
&= \phi(\widetilde{u}_{ij})\cdot\phi(\widetilde{S}_{\mu+1\,\mu+1}).
\end{aligned}$$

Let $\pi : Z[[t_1-1, \ldots, t_\mu-1]] \to Z[[t-1]]$ be given by $\pi(t_i-1) = t-1\ \forall i$. Also, let $\epsilon : Z[[t-1]] \to Z$ be the homomorphism which assigns to each power series its constant term; note that then $\epsilon\pi = \epsilon : Z[[t_1-1, \ldots, t_\mu-1]] \to Z$ and $\epsilon\pi\phi = \epsilon : Z[[t_1-1, \ldots, t_{\mu+1}-1]] \to Z$. The last equality of the preceding paragraph implies that

$$\epsilon\left[\frac{t^{2\mu x}-1}{t^2-1}\right] \cdot \epsilon(u_{ij}) = \epsilon(\widetilde{u}_{ij}) \cdot \epsilon\left[\frac{\pi\phi(\widetilde{\mathcal{S}}_{\mu+1\mu+1})}{t^2-1}\right].$$

Now

$$\pi\phi(\widetilde{\mathcal{S}}_{\mu+1\mu+1}) \equiv \sum_{i=1}^{\mu} \mu(i, \mu+1)\cdot(t^2-1) = \mu x\cdot(t^2-1)$$

(modulo $(t^2-1)^2$), and so

$$\epsilon\left[\frac{\pi\phi(\widetilde{\mathcal{S}}_{\mu+1\mu+1})}{t^2-1}\right] = \mu x .$$

Also,

$$\epsilon\left[\frac{t^{2\mu x}-1}{t^2-1}\right] = \mu x .$$

By cancellation, then, $\epsilon(u_{ij}) = \epsilon(\widetilde{u}_{ij}) = (-1)^{i+j}$. Q.E.D.

Note that Theorem (8.2) applies to ∇_L^{iso}, ∇_L^{conc}, ∇_L^{pli}, and ∇_L^{topi} as well, as all are multiples of ∇_L by power series in $t_1^2-1, \ldots, t_\mu^2-1$ with constant term 1.

For a particular choice of $i, j \in \{1, \ldots, \mu\}$, and any integers $r_1, \ldots, r_\mu \geq 0$, let $\gamma(r_1, \ldots, r_\mu)$ be the coefficient of $(t_1^2-1)^{r_1} \cdots (t_\mu^2-1)^{r_\mu}$ in $(-1)^{i+j}\det\mathcal{S}^{(ij)}/(t_j^2-1)$. Also, let $\delta(r_1, \ldots, r_\mu)$ be the greatest common divisor of those $\gamma(s_1, \ldots, s_\mu)$ with $s_k \leq r_k\ \forall k$ and $\Sigma s_k < \Sigma r_k$, and let $\gamma(r_1, \ldots, r_\mu)$ be the congruence class of $\gamma(r_1, \ldots, r_\mu)$ modulo $\delta(r_1, \ldots, r_\mu)$. Similarly, let $d(r_1, \ldots, r_\mu)$ be the greatest common divisor of those $c(s_1, \ldots, s_\mu)$ such that $s_k \leq r_k\ \forall k$ and $\Sigma\, s_k < \Sigma\, r_k$, and let $\bar{c}(r_1, \ldots, r_\mu)$ be the congruence class of $c(r_1, \ldots, r_\mu)$ modulo $d(r_1, \ldots, r_\mu)$. Then Theorem (8.2) immediately implies

COROLLARY (8.3). *For every choice of* $r_1, \ldots, r_\mu \geq 0$, $d(r_1, \ldots, r_\mu) = \delta(r_1, \ldots, r_\mu)$ *and* $\bar{c}(r_1, \ldots, r_\mu) = \bar{\gamma}(r_1, \ldots, r_\mu)$.

Note that consequently, the various $\delta(r_1, \ldots, r_\mu)$ and $\bar{\gamma}(r_1, \ldots, r_\mu)$ are independent of the choice of the particular i, j used in their definition; this yields many identities (of the forms $\delta_{ij} = \delta_{i'j'}$ and $\bar{\gamma}_{ij} = \bar{\gamma}_{i'j'}$) involving the integers $\mu(i_1, \ldots, i_p)$. It would be interesting to know if all of these are consequences of the relations known to hold between the $\bar{\mu}(i_1, \ldots, i_p)$ [26].

In the special case $\mu = 2$, Corollary (8.3) may be stated somewhat more succinctly. For $r_1, r_2 \geq 0$, let $[r_1 + 1, r_2 + 1] = (1, \ldots, 1, 2, \ldots, 2)$ be the sequence with $r_1 + 1$ ones and $r_2 + 1$ twos.

COROLLARY (8.4). *Suppose L is a two-component link. Then for every choice of $r_1, r_2 \geq 0$, $d(r_1, r_2) = \delta(r_1, r_2) = \Delta([r_1 + 1, r_2 + 1])$ and $\bar{c}(r_1, r_2) = \bar{\gamma}(r_1, r_2) = (-1)^{r_2}\bar{\mu}([r_1 + 1, r_2 + 1])$.*

The corresponding statement for the Alexander polynomial was originally discovered by Murasugi [30], in a somewhat weaker form, and sharpened by Smythe [38]. As the derivation of Corollary (8.4) from Corollary (8.3) has already appeared in the literature [38, 43], we will not repeat it here. Corollary (8.3) has analogous (though more complicated) consequences when $\mu > 2$; e.g., when $\mu = 3$, $c(1, 1, 1) \equiv \mu(1, 2, 3)^2 + \binom{\mu(1, 2)}{2}\binom{\mu(1, 3)}{2} + \binom{\mu(1, 2)}{2}\binom{\mu(2, 3)}{2} + \binom{\mu(1, 3)}{2}\binom{\mu(2, 3)}{2} \pmod{\Delta(1, 2, 3)}$. (The corresponding result for the Alexander polynomial is due to Murasugi [30].) We leave it as an exercise for the reader to deduce this, and also to use Torres' second relation to verify the following equalities: if $L = K_1 \cup K_2$ has linking number ℓ, then $c(0, 1) = c(1, 0)$ is the binomial coefficient $\binom{\ell}{2}$, $c(2, 0) = \binom{\ell}{3} + \ell a_2(K_1)$, and $c(0, 2) = \binom{\ell}{3} + \ell a_2(K_2)$.

COROLLARY (8.5). *The ± in Theorem 3 is unnecessary when $\mu = 2$.*

PROOF. This follows immediately from Corollary (8.4) and the fact that the $\bar{\mu}$-invariants are I-equivalence invariants [2]. Q.E.D.

Unfortunately, Corollary (8.3) cannot be used in a similar way, in general, as a link may have the property that its lowest-degree nonzero $c(r_1, \ldots, r_\mu)$ are related (through Corollary (8.3)) only to integers $\mu(i_1, \ldots, i_p)$ that have $\Delta(i_1, \ldots, i_p) = 1$.

Corollaries (8.3) and (8.4) apply equally well to the coefficients $c^{iso}(r_1, \ldots, r_\mu)$, $c^{conc}(r_1, \ldots, r_\mu)$, etc., since Theorem (8.2) applies to ∇^{iso}, etc. in place of ∇. In particular, concentrating our attention on ∇^{topi} we conclude from Corollary (8.3) that the combinations of $\bar{\mu}$-invariants corresponding to the various $\gamma(r_1, \ldots, r_\mu)$ can be lifted to integers that are invariant under extendable I-equivalences and invariant up to sign under arbitrary I-equivalences. The case $\mu = 2$ is particularly striking: in view of Corollary (8.5), we conclude that the various coefficients $c^{topi}(r_1, r_2)$ constitute a lifting of the various $(-1)^{r_2}\bar{\mu}([r_1 + 1, r_2 + 1])$ to integer invariants of topological I-equivalence, without any ambiguity of sign.

Similar considerations apply to the reduced potential function and its simplified versions. Theorem (8.2) implies directly that for every choice of $i, j \in \{1, \ldots, \mu\}$ there is a power series v_{ij} in $t^2 - 1$, with constant term $(-1)^{i+j}$, such that

$$t^q(t^2-1)\pi(\nabla_L) = v_{ij}\pi(det\,\mathcal{S}^{(ij)})$$

(where $q = \Sigma\, q_k$), namely $v_{ij} = \pi(u_{ij})$. Equivalently,

$$D_L = t^{-1-q}v_{ij}\,\pi(det\,\mathcal{S}^{(ij)});$$

since each of D_L^{iso}, D_L^{conc}, etc. is a multiple of D_L by a power series in t^2-1 whose constant term is 1, the same result applies to them as well.

In analogy with Corollary (8.3), we have

COROLLARY (8.6). *For $r \geq 0$ let γ_r be the sum of those $\gamma(r_1, \ldots, r_\mu)$ with $\Sigma\, r_k = r$. Then $\gamma_r = 0$ for $r < \mu - 2$, and $\gamma_r \equiv a_{r-\mu+2}$ (modulo $(a_0, \ldots, a_{r-\mu+3})$) for $r \geq \mu - 2$.*

The proof is similar to that of Proposition (7.3); we will not give it in detail.

In the particular case $\mu = 2$, we may "sum" instances of Corollary (8.4) to conclude

COROLLARY (8.7). *If $\mu = 2$ then for every $r \geq 0$*

$$a_r \equiv \sum_{s=0}^{r} (-1)^s \bar{\mu}([r-s+1,\ s+1])$$

(modulo the greatest common divisor of $\{\Delta([r+1, 1]), \ldots, \Delta([1, r+1])\}$).

Note that Corollary (8.7) is, generally, weaker than Corollary (8.6), because its indeterminacy is greater (that is, the greatest common divisor of $(a_0, \ldots, a_{r-\mu+3})$ is generally greater than that of $\{\Delta([r+1, 1]), \ldots, \Delta([1, r+1])\}$). Corollary (8.7) is stated by Cochran [3], who also deduces it from the theorem of Murasugi [30] and Smythe [38], though Cochran's statement has an additional minus sign, to compensate for the fact that he works with Conway's D_L rather than Hartley's.

9. The Arf invariant

In this section we are concerned with the Arf invariant, a mod 2 concordance invariant which is defined only for proper links. (A link $L = K_1 \cup \cdots \cup K_\mu$ is *proper* if $\mu = 1$, or if $\mu \geq 2$ and $\ell(K_i, L-K_i)$ is even for every i.)

For a knot, the Arf invariant is simply the congruence class (mod 2) of a_2 [16]; equivalently, it is the mod 2 congruence class of $(\Delta(-1)^2-1)/8$ [19]. For a proper link L of two or more components, $Arf(L)$ is defined to be the Arf invariant of any knot related to L. (Two links L_0 and L_1 are *related* if there is a smooth imbedding $r: D \hookrightarrow S^3 \times [0, 1]$ of a genus zero surface D, with boundary, such that $r(\partial D) = r(D) \cap (S^3 \times \{0, 1\})$, $r(D) \cap (S^3 \times \{0\}) = L_0 \times \{0\}$, $r(D) \cap (S^3 \times \{1\}) = L_1 \times \{1\}$, and the components of L_0 and L_1 are oriented compatibly with the orientation of D [32].) Ying-Qing Wu [47] has shown that for any link L (proper or not)

$$(9.1) \qquad a_2(L) \equiv \Sigma \, Arf(L') \pmod 2,$$

where the sum Σ is taken over the collection of all sublinks $L' \subseteq L$ such that L' and $L-L'$ are both proper links. (Here ϕ is considered to be a proper link, with $Arf(\phi) = 0$.)

Let $q(L)$ be 1 if L has only one component, and $q(L) = \Pi(-1 + \ell(K_i, L-K_i))$ $= \Pi\, q_i$ otherwise; observe that $q(L)$ is odd or even according to whether L is proper or not. Using this, we define an integer $A(L)$ as follows: $A(L) = a_2^{conc}(L)$ if L is a knot; and

$$A(L) = a_2^{conc}(L) + \sum_{L' \subset L} q(L')q(L-L')A(L')$$

if L has two or more components. (The sum on the right is to have a term for each nonempty sublink of L, other than L itself.)

THEOREM (9.2). *$A(L)$ is an integer concordance invariant. If L is proper then $A(L)$ reduces (mod 2) to $Arf(L)$, and if L is not proper then $A(L)$ is even.*

PROOF. Clearly a concordance between two links induces (by restriction) concordances between corresponding sublinks of the two links. That $A(L)$ is a concordance invariant follows immediately from this, the concordance invariance of linking numbers, and the concordance invariance of D^{conc}.

Suppose now that L is proper. If L has only one component, then $a_2(L) \equiv Arf(L) \pmod 2$; $a_2^{conc}(L) \equiv a_2(L) \pmod 2$ by Proposition (6.2), so $A(L) = a_2^{conc}(L)$ reduces (mod 2) to $Arf(L)$. Proceeding by induction on the number of components of L, we see that

$$A(L) \equiv a_2^{conc}(L) + \Sigma \, Arf(L') \pmod 2,$$

where the sum Σ is taken over the collection of all nonempty sublinks $L' \subset L$ such that L' and $L-L'$ are both proper. Since $a_2^{conc}(L) \equiv a_2(L) \pmod 2$, it follows from (9.1) that $A(L) \equiv Arf(L) \pmod 2$.

If L is not proper then still

$$A(L) \equiv a_2^{conc}(L) + \Sigma \, Arf(L') \pmod 2.$$

In this case, though, (9.1) implies that the sum $\Sigma \, Arf(L')$ is congruent (mod 2) to $a_2(L)$; since $a_2(L) \equiv a_2^{conc}(L) \pmod 2$, it follows that $A(L)$ is even. Q.E.D.

Murakami and Sugishita [29] have suggested a very different "lifting" of the Arf invariant for knots.

Suppose for the moment that the link L is "purely proper"; that is, every pair of components of L has even linking number. Since the linking numbers in L are all even, $a_0(L)$ is certainly even. A casual inspection of the matrix $\mathscr{S}$ shows that if $\mu \geq 4$, then every $\gamma(r_1, \ldots, r_\mu)$ with $\Sigma r_i \leq \mu$ is a combination of linking numbers, and hence is even; thus by Corollary (8.6), $a_2(L)$ is even. We immediately conclude

PROPOSITION (9.3). *If L is a purely proper link of $\mu \geq 4$ components, then $Arf(L)$ is simply the sum of the Arf invariants of the sublinks of L, other than L itself.*

Given a purely proper link L, we may apply Proposition (9.3) repeatedly, resulting in an expression of $Arf(L)$ in terms of the Arf invariants of the sublinks of L which have no more than three components. We leave it to the reader to verify that this repeated application of Proposition (9.3) suffices to prove

COROLLARY (9.4). *Suppose $L = K_1 \cup \cdots \cup K_\mu$ is a purely proper link of $\mu \geq 4$ components. Then the Arf invariant of L is*

$$\sum_{1 \leq i \leq \mu} \left(1 + \binom{\mu}{2}\right) Arf(K_i)$$
$$+ \sum_{1 \leq i < j \leq \mu} (\mu - 1) Arf(K_i \cup K_j)$$
$$+ \sum_{1 \leq i < j < k \leq \mu} Arf(K_i \cup K_j \cup K_k).$$

10. $c(1, 1)$ and the gordian problem

The *gordian number* $u(L)$ of a knot or link L is the least number of undercrossing-overcrossing reversals sufficient to transform some diagram of L into a diagram of a trivial link. For a two-component link of linking number $\ell = c(0, 0)$, the inequality $u(L) \geq |\ell|$ obviously holds. Boileau and Weber [1] have asked whether or not this inequality can be generalized to a relationship between the gordian number and the higher $\bar{\mu}$-invariants; in this section we discuss the extent to which the gordian number of a two-component link is related to the coefficient $c(1, 1)$ (and hence, by Corollary (8.4), to $\bar{\mu}(1, 1, 2, 2)$).

For $k \geq 0$ consider Rolfsen's link "3_k" [34], drawn in Figure 1; the diagram represented has $2k + 4$ crossings in all. (We are grateful to Sharon Richter for her drawings.) According to Rolfsen, 3_k has Alexander polynomial $k(t_1 - 1)(t_2 - 1)$, and hence potential function $\pm k(t_1 - t_1^{-1})(t_2 - t_2^{-1})$, so it has $c(1, 1) = \pm k$. As the mirror image of 3_k will have $c(1, 1) = \mp k$, we conclude

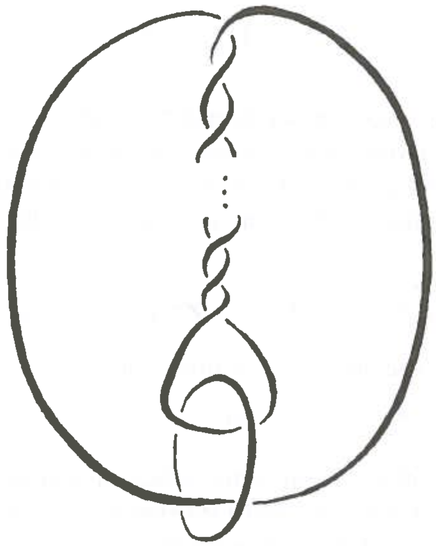

Figure 1

PROPOSITION (10.1). *For every* $c \in Z$ *there is a two-component link with* $c(1, 1) = c$ *and gordian number* ≤ 2.

Certainly then no analogue of the inequality $u(L) \geq |c(0, 0)|$ could apply to $c(1, 1)$. However, the situation is not completely hopeless; note that among the various links 3_k the only ones that seem to have gordian number ≤ 1 are 3_0 and 3_1. Also, notice that the single crossing to be reversed in trivializing 3_1 is not one of the lower four (which involve both components) but one of the upper two (which only involve a single component). This suggests that $c(1, 1)$ might be related to the reversing of crossings which only involve a single component, and indeed it is.

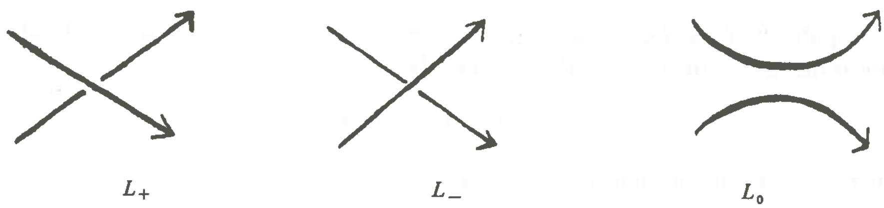

Figure 2

THEOREM (10.2). *Suppose that* L_+ *and* L_- *are two-component links, both with linking number* ℓ, *which possess diagrams that differ only at a single crossing,* *as indicated in Figure 2. Then for some* $m \in \mathbf{Z}$,

$$c_+(1,1) = c_-(1,1) + m \cdot (\ell - m).$$

PROOF. Note that since the linking numbers coincide, the crossing at which L_+ and L_- differ must involve only a single component; say $L_+ = K_1 \cup K_{2+}$ and $L_- = K_1 \cup K_{2-}$. Let $L_0 = K_1 \cup K_{20} \cup K_{30}$ be the three-component link obtained by smoothing the crossing in question, as indicated in Figure 2. Conway's first identity asserts that if

$$\sigma : \mathbf{Z}[t_1, t_2, t_3, t_1^{-1}, t_2^{-1}, t_3^{-1}] \to \mathbf{Z}[t_1, t_2, t_1^{-1}, t_2^{-1}]$$

is given by $\sigma(t_1) = t_1$ and $\sigma(t_2) = t_2 = \sigma(t_3)$, then

$$\nabla_{L_+} = \nabla_{L_-} + (t_2 - t_2^{-1}) \cdot \sigma(\nabla_{L_0}).$$

(This identity is most often applied to the reduced potential function, but the form we use here is that of Conway's original statement [5], and is easily justified by an argument along the lines of Hartley's proof of the reduced version [10].)

Let m be the linking number of K_1 and K_{20}; note that then $\ell - m$ is the linking number of K_1 and K_{30}. Let $q = q_1 = q_2 = -1 + \ell$; then we have

$$(10.3) \qquad t_1^q t_2^q \cdot \nabla_{L_+} = t_1^q t_2^q \cdot \nabla_{L_-} + t_1^q t_2^{q-1} \cdot (t_2^2 - 1)\, \sigma(\nabla_{L_0}).$$

The integers q_1, q_{20}, q_{30} corresponding to L_0 satisfy $q_1 = q$ and $q_{20} + q_{30} \not\equiv q$ (mod 2). Thus

$$t_1^q t_2^{q-1} \cdot \sigma(\nabla_{L_0}) = t_2^{2n} \cdot \sigma(t_1^{q_1} t_2^{q_{20}} t_3^{q_{30}} \cdot \nabla_{L_0})$$

for some integer n. Since the latter is a power series in $t_1^2 - 1$ and $t_2^2 - 1$ with constant term 0, (10.3) implies that

$$c_+(1,1) = c_-(1,1) + c_0(1,0,0).$$

By Corollary (8.3), $c_0(1,0,0) = \ell(K_1, K_{20})\,\ell(K_1, K_{30}) = m(\ell - m)$. Q.E.D.

Applying the analogous argument to the reduced potential function, it can be shown that in the situation of Theorem (10.2)

$$a_{2+} = a_{2-} + m(\ell - m) + \ell x,$$

where x is the linking number of K_{20} and K_{30}.

Since the trivial link has $c(1, 1) = 0$, an immediate consequence of Theorem (10.2) is

COROLLARY (10.4). *If a two-component link has gordian number* 1, *then either* *it has linking number* ± 1, *or else it has linking number* 0 *and* $|c(1, 1)|$ *a perfect square.*

(N.b. When the linking number of a two-component link is 0, the invariants $c(1, 1)$, a_2, and $-\bar{\mu}(1, 1, 2, 2)$ coincide.)

That any perfect square may occur in this situation is shown by the links L_k pictured in Figure 3. (For $k \geq 0$, the diagram drawn is to contain $4k + 2$ crossings, with $2k$ in each horizontal file of crossings.) The fact that L_k has $c(1, 1) = k^2$ follows immediately from the theorem, for if we consider the effect of modifying the uppermost crossing in $L_k = L_-$, L_+ is the trivial two-component link (so $c_+(1, 1) = 0$), and the three-component link L_0 has $\ell(K_1, K_{20}) = \pm k$.

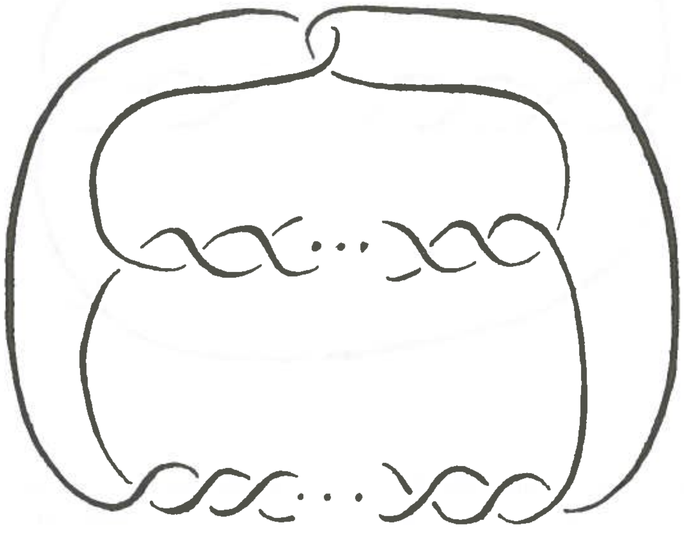

Figure 3

Corollary (10.4) implies that among the Alexander polynomials of two-component links of linking number 0, some are the Alexander polynomials of links of gordian number 1, and some are not. (Maeda and Murasugi [23] have also shown this, using very different techniques.) This contrasts with Kondo's theorem [18] that every knot shares its Alexander polynomial with some knot of gordian number 1.

More generally, Theorem (10.2) provides a necessary condition for a pair of two-component links of the same linking number to be related through a single crossing reversal (that is, to be separated by a "gordian distance" of one unit, in the sense of Murakami [28]).

Finally, we note the following interesting consequence of Theorem (10.2).

COROLLARY (10.5). *If a two-component link has odd linking number, its* $c(1, 1)$ *must be even.*

PROOF. According to Milnor [25], any two two-component links of the same linking number are homotopic; that is, one can be transformed into the other by a sequence of reversals of crossings, each of which only involves one component. By Theorem (10.2), then, if two two-component links have the same odd linking number, their $c(1, 1)$ are congruent (mod 2).

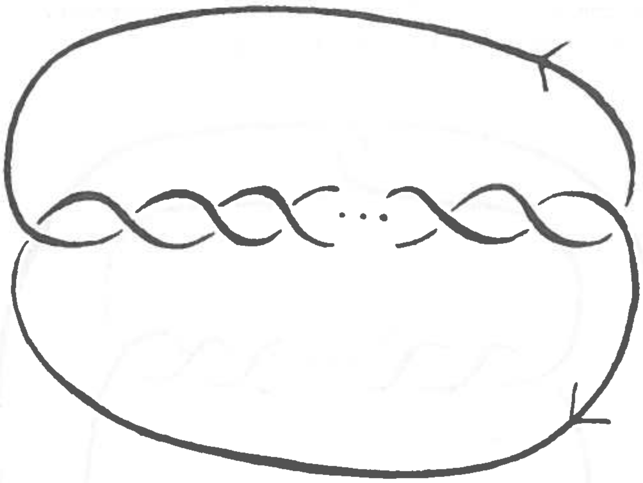

Figure 4

It suffices, then, to show that for any odd integer there is some two-component link with that linking number whose $c(1, 1)$ is even. To show this, consider the links L'_k pictured in Figure 4; for $k \geq 0$, the diagram drawn is to have $2k$ crossings. A simple induction using Conway's second identity [5] shows that the potential function of L'_k is a polynomial in $y = t_1 t_2 + t_1^{-1} t_2^{-1}$; moreover, it is a polynomial in odd powers of y when k is even, and a polynomial in even powers of y when k is odd.

When k is odd, L'_k will have even $q_1 = q_2$, and its potential function will be a polynomial in y^2. Note that if we express y^2 as a power series in $t_1^2 - 1$ and $t_2^2 - 1$, the constant term will be 4, the coefficients of the terms of degree one will be 0, and the coefficient of $(t_1^2 - 1)(t_2^2 - 1)$ will be 2. It follows immediately that the $c(1, 1)$ of L'_k is even.

This completes the proof in case the odd linking number being considered is positive. The analogous argument, with L'_k replaced by its mirror image, will apply when the linking number is negative. Q.E.D.

In contrast, there is no restriction on the integers that arise as $c(1, 1)$ for two-component links of even linking number, as Rolfsen's links 3_k and their mirror images show.

11. Examples

In this section we quickly survey the application of the results of the paper to some of the knots and two-component links tabulated by Conway [5].

As all knots are isotopic, the only simplified version of the potential function that's of interest when $\mu = 1$ is ∇^{conc} (or equivalently, D^{conc}). The Alexander polynomial is irreducible for all knots of up to seven crossings except 42, and hence for these knots $D = D^{conc}$ is a concordance invariant, which happens to distinguish them. (Of course, it is well known that the Fox-Milnor theorem [9] relating the Alexander polynomials of concordant knots serves to distinguish these from each other.) The "lifted Arf invariant" $a_2^{conc} = a_2$ distinguishes some of these knots (e.g., the trefoil and figure-eight have $a_2 = 1$ and -1, respectively), but not all (e.g., 2112 and 312 also have $a_2 = 1$ and -1, respectively).

Of the six two-component links on eight or fewer crossings with linking number 1, all but 3112 have irreducible Alexander polynomials. Indeed, 3112 has $\nabla^{conc} = 1$ (the same as 2, the Hopf link), so its potential function does not forbid its being concordant or p.l. I-equivalent to the Hopf link. The other four, though, are distinguished from these two (and from each other) by their $\nabla = \nabla^{pli}$, so there are at least five distinct p.l. I-equivalence classes among the six links. This is interesting because according to an unpublished result of Giffen, these six links (indeed, all links with linking number one) are topologically I-equivalent.

Conway lists ten two-component links of eight or fewer crossings with linking number 0. The non-trivial ones all have nonzero potential functions, so none of them is topologically I-equivalent to the trivial link. Several pairs of the non-trivial ones can be distinguished by their ∇^{topi} (e.g., 212 and 232), but several cannot. Among these latter, all can be distinguished by Theorem (5.1) (for instance, 212 and 21, 2, 2- have the same potential functions, but their components do not). Thus if any of these ten links are topologically I-equivalent, the I-equivalence must be so wild as to not be extendable.

Bibliography

[1] M. Boileau and C. Weber, *Le problème de J. Milnor sur le nombre gordien des noeuds algébriques*, Noeuds, Tresses, et Singularités (Monogr. Enseign. Math. 31), 49–98 (1983).

[2] A. J. Casson, *Link cobordism and Milnor's invariant*, Bull. London Math. Soc. 7, 39–40 (1975).

[3] T. D. Cochran, *Concordance invariance of coefficients of Conway's link polynomial*, Invent. Math. 82, 527–541 (1985).
[4] T. D. Cochran, *Derivatives of links, Milnor's concordance invariants and Massey products*, preprint, Univ. of California, Berkeley, Calif. (1986).
[5] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra, Oxford, Pergamon Press, 329–358 (1969).
[6] R. H. Crowell, *Nonalternating links*, Illinois J. Math. 3, 101–120 (1959).
[7] R. H. Fox, *Free differential calculus*. I, Ann. of Math. (2) 57, 547–560 (1953).
[8] R. H. Fox, *A quick trip through knot theory*, Topology of 3-Manifolds and Related Topics, Englewood Cliffs, N. J., Prentice-Hall, 120–167 (1962).
[9] R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. 3, 257–267 (1966).
[10] R. Hartley, *The Conway potential function for links*, Comm. Math. Helv. 58, 365–378 (1983).
[11] J. A. Hillman, *Alexander Ideals of Links*, Berlin, Heidelberg, and New York, Springer-Verlag (1981).
[12] R. Holmes and N. Smythe, *Algebraic invariants of isotopy of links*, Amer. J. Math. 88, 646–654 (1966).
[13] F. Hosokawa, *On ∇-polynomials of links*, Osaka Math. J. 10, 273–282 (1958).
[14] J. Hoste, *The Arf invariant of a totally proper link*, Topology Appl. 18, 163–177 (1984).
[15] J. Hoste, *The first coefficient of the Conway polynomial*, Proc. Amer. Math. Soc. 95, 299–302 (1985).
[16] L. H. Kauffman, *The Conway polynomial*, Topology 20, 101–108 (1981).
[17] A. Kawauchi, *On the Alexander polynomials of cobordant links*, Osaka J. Math. 15, 151–159 (1978).
[18] H. Kondo, *Knots of unknotting number 1 and their Alexander polynomials*, Osaka J. Math. 16, 551–559 (1979).
[19] J. P. Levine, *Polynomial invariants of knots of codimension two*, Ann. of Math. (2) 84, 534–554 (1966).
[20] J. P. Levine, *Localization of link modules*, Low Dimensional Topology (Contemp. Math. 20), Providence, R. I., Amer. Math. Soc., 213–229 (1983).
[21] J. P. Levine, *Surgery on links and the $\bar{\mu}$-invariants*, Topology 26, 45–61 (1987).
[22] J. P. Levine, *An approach to homotopy classification of classical links*, Trans. Amer. Math. Soc. 306, 361–387 (1988).
[23] T. Maeda and K. Murasugi, *Covering linkage invariants and Fox's problem 13*, Low Dimensional Topology (Contemp. Math. 20), Providence, R. I., Amer. Math. Soc., 271–283 (1983).
[24] W. S. Massey, *Completion of link modules*, Duke Math. J. 47, 399–420 (1980).
[25] J. Milnor, *Link groups*, ann. of Math. (2) 59, 177–195 (1954).
[26] J. Milnor, *Isotopy of links*, Algebraic Geometry and Topology, Princeton, N.J., Princeton Univ. Press, 280–306 (1957).
[27] H. Murakami, *The Arf invariant and the Conway polynomial of a link*, Math. Sem. Notes Kobe Univ. 11, 335–344 (1983).
[28] H. Murakami, *Some metrics on classical knots*, Math. Ann. 270, 35–45 (1985).
[29] H. Murakami and K. Sugishita, *Triple points and knot cobordism*, Kobe J. Math. 1, 1–16 (1984).
[30] K. Murasugi, *On Milnor's invariant for links*, Trans. Amer. Math. Soc. 124, 94–110 (1966).

[31] Y. Nakagawa, *On the Alexander polynomials of slice links*, Osaka J. Math. 15, 151–159 (1978).
[32] R. A. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. 18, 543–555 (1965).
[33] D. Rolfsen, *Isotopy of links in codimension two*, J. Indian Math. Soc. 36, 263–278 (1972).
[34] D. Rolfsen, *Localized Alexander invariants and isotopy of links*, Ann. of Math. (2) 101, 1–19 (1975).
[35] D. Rolfsen, *Knots and Links*, Berkeley, Calif., Publish or Perish (1976).
[36] D. Rolfsen, *Piecewise-linear I-equivalence of links*, Low Dimensional Topology, Cambridge, Cambridge Univ. Press, 161–178 (1985).
[37] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. 110, 571–592 (1934).
[38] N. Smythe, *Isotopy invariants of links and the Alexander matrix*, Amer. J. Math. 89, 693–704 (1967).
[39] J. Stallings, *Homology and central series of groups*, J. Alg. 2, 170–181 (1965).
[40] D. Stein, *Massey products in the cohomology of groups with applications to link theory*, Ph. D. Dissertation, Brandeis Univ., Waltham, Mass. (1986).
[41] G. Torres, *On the Alexander polynomial*, Ann. of Math. (2) 57, 57–89 (1953).
[42] G. Torres and R. H. Fox, *Dual presentations of the group of a knot*, Ann. of Math. (2) 59, 211–218 (1954).
[43] L. Traldi, *Milnor's invariants and the completions of link modules*, Trans. Amer. Math. Soc. 284, 401–424 (1984).
[44] L. Traldi, *On the Goeritz matrix of a link*, Math. Z. 188, 203–213 (1985).
[45] A. G. Tristram, *Some cobordism invariants for links*, Math. Proc. Cambr. Phil. Soc. 66, 251–264 (1969).
[46] V. G. Turaev, *Reidemeister torsion in knot theory*, Uspekhi Mat. Nauk. 41, 98–147 (1986) (English translation Russian Math. Surveys 41, 119–182 (1986)).
[47] Yinq-Qing Wu, *On the Arf invariant of links*, Math. Proc. Cambr. Phil. Soc. 100, 355–359 (1986).

Department of Mathematics
Lafayette College
Easton, Pennsylvania 18042
U.S.A.