Tied dice. II. Some asymptotic results

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Abstract

A *dice family* $D(n, a, b, s)$ includes all lists $(x_1, ..., x_n)$ of integers with $n \geq 1$, $a \leq x_1 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. Given two dice X and Y we compare the number of pairs (i, j) with $x_i < y_j$ to the number of pairs (i, j) with $x_i > y_j$. If the second number is larger then X is *stronger* than Y, and if the two numbers are equal then X and Y are *tied*. In previous work it has been observed that the density of ties in $D(n, a, b, s)$ is generally lower than one might expect. In this note we provide more information about this observation by calculating the asymptotic proportion of ties in certain kinds of dice families. Many other properties of dice families remain to be determined.

1. Introduction

The fact that nonstandard dice can yield nontransitive results was noted by Efron, and popularized by Gardner [1]. In the decades since then, nontransitivity and other surprising characteristics of nonstandard dice have attracted a good deal of attention. Technical references are given at the end of the paper; many less technical discussions may also be found on the internet.

Definition 1.1. *A (generalized)* die *is a list* $X = (x_1, ..., x_n)$ *of integers such that* $x_1 \leq x_2 \leq ... \leq x_n$.

Definition 1.2. *If* X *and* Y *are dice then* X *is* stronger *than* Y*, denoted* $X > Y$*, if there are strictly more ordered pairs* (x_i, y_j) *with* $x_i > y_j$ *than with* $x_i < y_j$*. If* $X \not> Y$ *and* $Y \not> X$ *then* X *and* Y *are* tied*, denoted* $X \sim Y$.

Definition 1.3. *If $a \leq b, n \geq 1$ and s are integers then $D(n,a,b,s)$ denotes the* dice family *consisting of all dice $(x_1,...,x_n)$ with $\sum x_i = s$ and $a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b$.*

When $n \geq 3$ both *stronger* and *tied* are nontransitive in general. For example,

$$(1,1,4,5,5,5) > (3,3,3,3,3,6) > (1,1,1,6,6,6) > (1,1,4,5,5,5).$$

In addition, all three of these dice tie the standard die, $(1,2,3,4,5,6)$; indeed, $(1,2,3,4,5,6)$ is the unique element of $D(6,1,6,21)$ that ties all dice in this family [6].

Dice families were introduced in [6], and studied further in [3, 7, 9]. The general theme of these papers is this: As all dice in $D(n,a,b,s)$ have the same mean label value, one might expect that most pairs of dice in $D(n,a,b,s)$ are tied. Instead, ties are relatively rare.

For instance, Table 1 indicates that the proportion of tied pairs varies considerably among the dice families $D(6,1,6,s)$ with more than one element; but it is never as much as 30%.

s	*wins*	*ties*	s	*wins*	*ties*	s	*wins*	*ties*
8	1	0	9	3	0	10	9	1
11	20	1	12	33	12	13	62	4
14	103	17	15	148	23	16	216	37
17	266	34	18	289	117	19	369	66
20	396	100	21	371	125	22	396	100
23	369	66	24	289	117	25	266	34
26	216	37	27	148	23	28	103	17
29	62	4	30	33	12	31	20	1
32	9	1	33	3	0	34	1	0

Table 1: Wins and ties in $D(6,1,6,s)$, $8 \leq s \leq 34$.

Taken together, these dice families include 458 dice. A particularly striking instance of the "ties are rare" theme is the fact that only three of these 458 dice tie all the others in their families: $(1,1,3,3,5,5)$ ties all the elements of $D(6,1,6,18)$, $(1,2,3,4,5,6)$ ties all the elements of $D(6,1,6,21)$ and $(2,2,4,4,6,6)$ ties all the elements of $D(6,1,6,24)$. We call such dice *balanced*; a simple characterization is given in [6].

Another instance of the "ties are rare" theme is the Tied Dice Theorem of [3]. This theorem states that if $G(n,a,b,s)$ is the simple graph with $V(G(n,a,b,s)) = D(n,a,b,s)$ and $E(G(n,a,b,s)) = \{XY \mid X > Y$ or $Y > X\}$ then aside from isolated vertices, $G(n,a,b,s)$ is connected and has diameter ≤ 3. In fact, the diameter is ≤ 2 unless $n = 3$, $b = a + 8$ and $s = 3a + 12$; even then $G(n,a,b,s)$ has only one pair of vertices that do not share a neighbor.

Many open questions about the dice families $D(n,a,b,s)$ and the associated graphs $G(n,a,b,s)$ are mentioned in [7, 9]. Here are some of them:

- Is it possible to give simple characterizations of dice that are not weaker than any other dice in the same family, or not stronger?
- What can one say about the structure of the directed graph obtained from $G(n,a,b,s)$ by using *stronger* to direct edges?
- The Tied Dice Theorem suggests that the graphs $G(n,a,b,s)$ are highly connected. Are there results that confirm this suggestion?
- What can be said about asymptotic properties of $D(n,a,b,s)$?

The purpose of this note is to present some answers to the last question. Examples indicate that in general, ties are most common in dice families $D(n,a,b,s)$ with $\frac{s}{n} \approx \frac{a+b}{2}$. (See Table 1, for instance.) Consequently it is natural to begin studying the asymptotic behavior of the proportion of ties with the dice families $D(n,a,b,\frac{n}{2}(a+b))$. Observe that the function $(x_1,...,x_n) \mapsto (x_1 - a + 1, ..., x_n - a + 1)$ defines a *stronger*-preserving bijection between $D(n,a,b,\frac{n}{2}(a+b))$ and $D(n,1,b-a+1,\frac{n}{2}(b-a+2))$, so we lose no generality by restricting attention to dice families $D(n,1,k,\frac{(k+1)n}{2})$; $(k+1)n$ must be even of course.

Theorem 1.4. *For every fixed, odd integer $n \geq 3$ the proportion of ties in $D(n,1,k,\frac{(k+1)n}{2})$ approaches 0 as $k \to \infty$. (Note that k must be odd.)*

Theorem 1.5. *For every fixed, even integer $n \geq 2$ the proportion of ties in $D(n,1,k,\frac{(k+1)n}{2})$ has a positive lower limit as $k \to \infty$.*

Theorem 1.6. *For every fixed integer $k \geq 4$, the proportion of ties in $D(n,1,k,\frac{(k+1)n}{2})$ approaches 0 as $n \to \infty$. (Note that if k is even then n must also be even.)*

Theorems 1.4, 1.5 and 1.6 certainly do not provide a complete theory. In particular, it would be interesting to evaluate the precise value of the positive lower limit in Theorem 1.5; some empirical results are mentioned below. More generally, it would be good to precisely describe the observed drop-off in the proportion of ties when $\frac{s}{n}$ is far from $\frac{a+b}{2}$.

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2. Theorems 1.4 and 1.5

Proof of Theorem 1.4: Suppose $n \geq 3$ is odd. Then n^2 (the number of rolls of two dice) is odd, so two dice that share no labels cannot tie. As k grows, it becomes increasingly rare for two dice to share even one label, so typical pairs of dice do not tie. $\Box$

Proof of Theorem 1.5: All dice in $D(2, 1, k, k+1)$ are tied, so when $n = 2$ the limit mentioned in the theorem is 1.

Suppose $n \geq 4$ is even. If k is large then a typical die in $D(n, 1, k, \frac{(k+1)n}{2})$ has n distinct labels, fairly evenly spaced. On average, then, roughly $(1/(n+1))^n$ of the rest of the dice in $D(n, 1, k, \frac{(k+1)n}{2})$ have all their labels lying in the interval between the typical die's two middle labels. These other dice tie the typical one, so the proportion of ties is $\gtrsim (1/(n+1))^n$. $\Box$

n \ k	8	9	10	20	30	40	50	60	80	100	200	300
4	.37	.37	.35	.34	.36	.37	.38	.39	.41	.42	.44	.45
6	.20	.22	.18	.18	.18	.19	.20	.21				
8	.13	.13	.12	.11	.12							
10			.08	.08								

Table 2: The proportion of ties among non-balanced dice in $D(n, 1, k, \frac{n}{2}(k+1))$.

We do not know how to find precise values for the lower limits of tying proportions in Theorem 1.5. Computer results given in Table 2 indicate that $(1/(n+1))^n$ is an underestimate; $(1/2)^{\frac{n-2}{2}}$ seems closer. (To avoid wasting run time, the programs that generated the data displayed in Table 2 excluded balanced dice from consideration. This exclusion has little effect, as there are few balanced dice in $D(n,1,k,\frac{(k+1)n}{2})$ when $k \leq 2n+1$, and none when $k > 2n+1$ [6].) Whatever the precise values may be, the table certainly suggests that the asymptotic tying proportion decreases as n increases. If true, this would explain why $n = 4$ is the most difficult case of the Tied Dice Theorem [3]: it is hard to find non-tied pairs when $n = 4$ because there are so few of them!

3. Theorem 1.6

It is a simple matter to explain why Theorem 1.6 requires $k \geq 4$: when $k \leq 3$, the proportion of ties in $D(n,1,k,\frac{(k+1)n}{2})$ is 1. For $k = 1$ this is obvious, as $D(n,1,1,n)$ has only one element. Similarly, if n is even then $D(n,1,2,\frac{3n}{2})$ has only one element.

When $k = 3$, $D(n,1,k,\frac{(k+1)n}{2}) = D(n,1,3,2n)$ has more than one element; however the proportion of ties is still 1. Note that there is a bijective correspondence between $\{0,...,\lfloor\frac{n}{2}\rfloor\}$ and $D(n,1,3,2n)$ under which an integer x corresponds to the die

$$X = (\underbrace{1,...,1}_{x},\underbrace{2,...,2}_{n-2x},\underbrace{3,...,3}_{x}).$$

If

$$Y = (\underbrace{1,...,1}_{y},\underbrace{2,...,2}_{n-2y},\underbrace{3,...,3}_{y})$$

is any other element of $D(n,1,3,2n)$ then the n^2 rolls of X against Y include $x(n-y)$ wins for X when it rolls a 3, $y(n-x)$ wins for Y when it rolls a 3, $(n-2x)y$ wins for X when it rolls a 2 and $(n-2y)x$ wins for Y when it rolls a 2. X and Y are tied because

$$x(n-y) + (n-2x)y = y(n-x) + (n-2y)x.$$

Two definitions will be useful in our proof of Theorem 1.6.

Definition 3.1. *The* characteristic vector *of a die* $X = (x_1,...,x_n) \in D(n,1,k,\frac{(k+1)n}{2})$ *is the vector* $v^X = (v_1^X,...,v_k^X)$ *with*

$$v_i^X = |\{j \in \{1,...,n\} \mid x_j = i\}|.$$

Observe that the characteristic vectors of elements of $D(n,1,k,\frac{(k+1)n}{2})$ are the vectors $(v_1^X,...,v_k^X)$ whose coordinates are non-negative integers such that

$$\sum_{i=1}^{k} v_i^X = n \text{ and } \sum_{i=1}^{k} i v_i^X = \frac{(k+1)n}{2}.$$

Definition 3.2. *The* normalized characteristic vector *of X is $w^X = \frac{1}{n}v^X$.*

To begin the proof of Theorem 1.6, consider a fixed die $A \in D(n, 1, k, \frac{(k+1)n}{2})$. The normalized characteristic vectors of elements of $D(n, 1, k, \frac{(k+1)n}{2})$ are the vectors $(w_1^X, ..., w_k^X)$ whose coordinates are non-negative rational numbers $\frac{m}{n}$ such that

$$\sum_{i=1}^{k} w_i^X = 1 \text{ and } \sum_{i=1}^{k} i w_i^X = \frac{k+1}{2}.$$

As $A \in D(n, 1, k, \frac{(k+1)n}{2})$, w^A also satisfies these conditions. Consequently the normalized characteristic vectors of elements of $D(n, 1, k, \frac{(k+1)n}{2})$ are the vectors $w = (w_1, ..., w_k)$ whose coordinates are non-negative rational numbers $\frac{m}{n}$ such that $w \cdot (1, ..., 1) = w^A \cdot (1, ..., 1)$ and $w \cdot (1, 2, 3, ..., k) = w^A \cdot (1, 2, 3, ..., k)$.

Also, a die $X \in D(n, 1, k, \frac{(k+1)n}{2})$ ties A if and only if $v^A = (v_1^A, ..., v_k^A)$ and $v^X = (v_1^X, ..., v_k^X)$ satisfy the equality

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} v_i^X v_j^A = \sum_{i=2}^{k} \sum_{j=1}^{i-1} v_i^X v_j^A.$$

(The formula expresses the fact that each of the two dice wins the same number of rolls.) Dividing by n and collecting terms, this equality becomes

$$\sum_{i=1}^{k} \left(\sum_{j=i+1}^{k} v_j^A - \sum_{j=1}^{i-1} v_j^A \right) w_i^X = 0.$$

Equivalently, if we let u^A denote the vector

$$\left(\sum_{j=2}^{k} v_j^A, \sum_{j=3}^{k} v_j^A - \sum_{j=1}^{1} v_j^A, \sum_{j=4}^{k} v_j^A - \sum_{j=1}^{2} v_j^A, ..., \sum_{j=k}^{k} v_j^A - \sum_{j=1}^{k-2} v_j^A, - \sum_{j=1}^{k-1} v_j^A \right)$$

then X ties A if and only if $w^X \cdot u^A = 0$. As A ties itself, $w^A \cdot u^A = 0$; hence X ties A if and only if $w^X \cdot u^A = w^A \cdot u^A$.

We see that the question "Among the dice in $D(n, 1, k, \frac{(k+1)n}{2})$, what proportion tie A?" is the same as the question "Among the points $w = (w_1, ..., w_k) \in [0, 1]^k$ whose coordinates are rational numbers $\frac{m}{n}$ such that $w \cdot (1, ..., 1) = w^A \cdot (1, ..., 1)$ and $w \cdot (1, 2, 3, ..., k) = w^A \cdot (1, 2, 3, ..., k)$, what proportion also satisfy $w \cdot u^A = w^A \cdot u^A$?"

To answer this question, notice first that the set S of points $w = (w_1, ..., w_k) \in [0,1]^k$ with $w \cdot (1, ..., 1) = w^A \cdot (1, ..., 1)$ and $w \cdot (1, 2, 3, ..., k) = w^A \cdot (1, 2, 3, ..., k)$ is of dimension $k-2$. The dimension of S cannot be more than $k-2$ because the equations $w \cdot (1, ..., 1) = w^A \cdot (1, ..., 1)$ and $w \cdot (1, 2, 3, ..., k) = w^A \cdot (1, 2, 3, ..., k)$ are linearly independent; and the dimension of S cannot be less than $k-2$ because every $(w_1, ..., w_{k-2}) \in [0, \frac{1}{k^2}]^{k-2}$ extends to a point $(w_1, ..., w_k) \in S$.

As $n \to \infty$, the proportion of points $w = (w_1, ..., w_k) \in [0,1]^k$ with coordinates of the form $\frac{m}{n}$ that lie in S and satisfy $w \cdot u^A = w^A \cdot u^A$ limits to the fraction of the $(k-2)$-dimensional volume of S that lies in the subset T of S defined by the equation $w \cdot u^A = w^A \cdot u^A$. If u^A is not a linear combination of the vectors $(1, ..., 1)$ and $(1, 2, 3, ..., k)$, then T is of dimension $k-3$ and this fraction is 0. In order to complete the proof of Theorem 1.6, then, it suffices to explain why a typical die $A \in D(n, 1, k, \frac{(k+1)n}{2})$ has the property that u^A is not a linear combination of $(1, ..., 1)$ and $(1, 2, 3, ..., k)$.

Lemma 3.3. *The subspace of $\mathbb{R}^k$ spanned by $(1, ..., 1)$ and $(1, 2, 3, ..., k)$ consists of all vectors $(u_1, ..., u_k)$ with the property that $u_{i+1} - u_i$ does not vary with i.*

Proof. There are $k-2$ independent equations

$$u_{i+1} - u_i = u_k - u_{k-1}$$

with $1 \leq i \leq k-2$, and these equations are all satisfied by $(1, ..., 1)$ and $(1, 2, 3, ..., k)$. $\square$

Note that if $A \in D(n, 1, k, \frac{(k+1)n}{2})$ and $u^A = (u_1^A, ..., u_k^A)$ then

$$u_{i+1}^A - u_i^A = -(v_i^A + v_{i+1}^A),$$

so Lemma 3.3 tells us that u^A is a linear combination of the two vectors $(1, ..., 1)$ and $(1, 2, 3, ..., k)$ if and only if the sums $v_i^A + v_{i+1}^A$ do not vary with i. The defining equations of $D(n, 1, k, \frac{(k+1)n}{2})$, namely

$$\sum_{i=1}^{k} v_i^A = n \text{ and } \sum_{i=1}^{k} i v_i^A = \frac{(k+1)n}{2},$$

are clearly independent of equations of the form

$$v_i^A + v_{i+1}^A = v_j^A + v_{j+1}^A,$$

where $i \neq j$. Consequently for a typical $A \in D(n, 1, k, \frac{(k+1)n}{2})$, u^A is not a linear combination of $(1, ..., 1)$ and $(1, 2, 3, ..., k)$.

This completes the proof of Theorem 1.6. $\square$

By the way, one should not be surprised by the fact that if u^A is a linear combination of $(1, ..., 1)$ and $(1, 2, 3, ..., k)$, then A ties an unusually large proportion of the elements of $D(n, 1, k, \frac{(k+1)n}{2})$. Lemma 3.3 and Theorem 2 of [6] tell us that in fact such an A is balanced, i.e., it ties *all* the elements of $D(n, 1, k, \frac{(k+1)n}{2})$.

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