Rearrangements of generalized shellings

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Abstract

We discuss two generalizations of the Rearrangement Lemma for Shellings.

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1. Introduction

A *complex* Δ on a finite set E is a family of subsets of E which contains all singleton subsets and has the property that whenever $S \in \Delta$ and $T \subseteq S \subseteq E$, $T \in \Delta$ too. The elements of Δ are *faces*, and the maximal elements are *facets*. An ordering $F_1, ..., F_t$ of the facets of Δ is a *shelling* if there are subsets $\mathfrak{R}(F_1) \subseteq F_1, ..., \mathfrak{R}(F_t) \subseteq F_t$ such that the Boolean intervals $[\mathfrak{R}(F_i), F_i]$ partition Δ and $\mathfrak{R}(F_i) \nsubseteq F_j$ for $i > j$; that is, every subset of F_j appears in exactly one interval $[\mathfrak{R}(F_i), F_i]$ with $i \leq j$ and in no interval $[\mathfrak{R}(F_i), F_i]$ with $i > j$. (Recall that a *Boolean interval* is $[A, B] = \{S \mid A \subseteq S \subseteq B\}$.) The subsets $\mathfrak{R}(F_1), ..., \mathfrak{R}(F_t)$ are the *restrictions* of $F_1, ..., F_t$. Observe that there is no freedom in choosing them: $\mathfrak{R}(F_1)$ must be $\emptyset$ and for $j > 1$, $\mathfrak{R}(F_j)$ must be the intersection of the subsets of F_j which are not contained in any F_i with $i < j$. For $F_1, ..., F_t$ to be a shelling it must be that for every $j > 1$, this intersection itself is not contained in any F_i with $i < j$. Shellings of *pure* complexes (those whose facets are all of the same cardinality) have been studied for several decades, resulting in a body of theory that is both beautiful and useful; see [1, 2, 5, 7] for four different discussions.

In their work introducing shellings of non-pure complexes [3], Björner and Wachs proved the Rearrangement Lemma: a shelling $F_1, ..., F_t$ may be rearranged as $F'_1, ..., F'_t$ so that $|F'_a| \geq |F'_{a+1}|$ for each a, and moreover the two shellings have the same restriction map $\mathfrak{R}$, i.e., $\mathfrak{R}(F_i) = \mathfrak{R}(F'_j)$ when $F_i = F'_j$. In the present note we discuss versions of the Rearrangement Lemma for two types of generalized shellings.

The first type of generalized shelling we consider is an *$\mathcal{S}$-partition* [4]. This is a list $F_1, ..., F_t$ which satisfies all the requirements for a shelling except that not all the F_i are required to be facets. Observe that every complex Δ admits $\mathcal{S}$-partitions; for instance any list $F_1, ..., F_t$ of all the faces of Δ, listed in any order of nondecreasing cardinality, is an $\mathcal{S}$-partition. The existence of these trivial examples may give the impression that $\mathcal{S}$-partitions are so ubiquitous as to be uninteresting, but it turns out that every $\mathcal{S}$-partition of a complex Δ gives rise to a shelling of a related complex Δ' through the following simple construction. Given a list $F_1, ..., F_t$ of subsets of E such that $F_i \nsubseteq F_j$ for $i > j$, let $e'_1, ..., e'_{t-1} \notin E$, and let $F'_1 = F_1 \cup \{e'_1, ..., e'_{t-1}\}$, $F'_2 = F_2 \cup \{e'_2, ..., e'_{t-1}\}$, ..., $F'_{t-1} = F_{t-1} \cup \{e'_{t-1}\}$, $F'_t = F_t$. Let Δ be the complex of subsets of E which are contained in some F_i, and let Δ' be the complex of subsets of $E' = E \cup \{e'_1, ..., e'_{t-1}\}$ which are contained in some F'_i. Then it is easily seen that $F_1, ..., F_t$ is an $\mathcal{S}$-partition of Δ if and only if $F'_1, ..., F'_t$ is a shelling of Δ'. (The elements $e'_1, ..., e'_{t-1}$ are adjoined to E to guarantee that no F'_j contains another, so there is a more economical version of the construction which introduces a new element e'_j only if there is an $i > j$ with $F_j \subseteq F_i$.)

The Rearrangement Lemma for Shellings generalizes to the following.

The Rearrangement Lemma for $\mathcal{S}$-Partitions. *An $\mathcal{S}$-partition $F_1, ..., F_t$ may be rearranged as an $\mathcal{S}$-partition $F'_1, ..., F'_t$ with the same restriction function, such that for each a either $|F'_a| \geq |F'_{a+1}|$ or $F'_a = F'_{a+1} - \{x\}$ for some $x \in F'_{a+1}$.*

Shellings and $\mathcal{S}$-partitions have several useful applications in network reliability. One application involves algorithms that produce *sums of disjoint products* (or *SDP*); see [6] for an exposition. Classical SDP algorithms produce lists of disjoint products which define $\mathcal{S}$-partitions when the products are suitably interpreted as Boolean intervals. A generalization of classical disjoint product algorithms involves *multiple-variable inversion* (*MVI*). We call our second type of generalized shelling an *MVI $\mathcal{S}$-partition* because the lists of disjoint products produced by MVI-SDP algorithms satisfy the definition, when the products are suitably interpreted as collections of sets. We hope that a structural theory of $\mathcal{S}$-partitions and MVI $\mathcal{S}$-partitions may be useful in analyzing the performance of the two types of SDP algorithms.

An *MVI interval* of sets is determined by a set F and a family $\mathfrak{R}$ of pairwise disjoint subsets of F; the interval is $[\mathfrak{R}; F] = \{$ subsets of F which intersect every element of $\mathfrak{R}$ $\}$. MVI intervals generalize Boolean intervals: $[\emptyset; F] = [\emptyset, F]$ and if the elements of $\mathfrak{R}$ are all singletons then $[\mathfrak{R}; F] = [\{r \mid \{r\} \in \mathfrak{R}\}, F]$. An *MVI $\mathcal{S}$-partition* of a complex Δ is a list $F_1, ..., F_t$ of faces of Δ together with a *restriction function* $\mathfrak{R}$ such that for each i $[\mathfrak{R}(F_i); F_i]$ is an MVI interval, these MVI intervals partition Δ, and for each j every subset of F_j appears in an MVI interval $[\mathfrak{R}(F_i); F_i]$ with $i \leq j$. As with ordinary $\mathcal{S}$-partitions, there is no freedom in choosing $\mathfrak{R}(F_j)$: $\mathfrak{R}(F_1)$ must be $\emptyset$ and for $j > 1$, $\mathfrak{R}(F_j)$ must be $\{$ minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}\}$; in order for $\mathfrak{R}(F_j)$ to be an MVI interval these minimal relative complements must be nonempty and pairwise disjoint. (N.b. Requiring the distinct minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ to be pairwise disjoint does not forbid a minimal relative complement from occurring more than once among $F_j - F_1, ..., F_j - F_{j-1}$.)

The Rearrangement Lemma for MVI $\mathcal{S}$-Partitions. *An MVI $\mathcal{S}$-partition $F_1, ..., F_t$ may be rearranged as an MVI $\mathcal{S}$-partition $F'_1, ..., F'_t$ with the same restriction function, such that for each a either $|F'_a| \geq |F'_{a+1}|$ or $F'_a \cap F'_{a+1} \in [\mathfrak{R}(F'_a); F'_a]$.*

An *MVI shelling* of a complex is an MVI $\mathcal{S}$-partition which involves only facets. Considering the Rearrangement Lemma for Shellings, one might conjecture that MVI shellings can always be rearranged to be nonincreasing with respect to cardinality. This turns out not to be the case. For instance, the complex with facets $\{c, d, f, g, i\}$, $\{a, b, e, h, i\}$, and $\{b, d, e, f, g, h\}$ is MVI shelled by the given order but is not MVI shelled by any order beginning with $\{b, d, e, f, g, h\}$; whichever of $\{c, d, f, g, i\}$, $\{a, b, e, h, i\}$ appears second in such an order will have non-disjoint minimal relative complements. (The smaller facets $\{g, h, i\}$, $\{e, f, i\}$ and $\{a, c\}$ may be included to produce an example that is not subject to series or parallel reductions.)

2. Proofs

We begin with useful alternative definitions of the two types of $\mathcal{S}$-partitions, generalizing a standard alternative definition of shellings.

Proposition 2.1. *A list $F_1, ..., F_t$ is an MVI $\mathcal{S}$-partition if and only if for every $j > 1$, the distinct minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ are nonempty and pairwise disjoint. Moreover, if $j > 1$ then $\mathfrak{R}(F_j)$ is precisely the collection of these distinct minimal relative complements.*

Proof. Let $F_1, ..., F_t$ be an MVI $\mathcal{S}$-partition with restrictions $\mathfrak{R}(F_j)$. We claim that if $i < j$ then $F_j - F_i$ contains an element of $\mathfrak{R}(F_j)$. Suppose not; then $F_j - (F_j - F_i) = F_i \cap F_j$ intersects every element of $\mathfrak{R}(F_j)$, so $F_i \cap F_j \in [\mathfrak{R}(F_j); F_j]$. This is impossible, though, for $F_i \cap F_j$ is a subset of F_i and hence must appear in some $[\mathfrak{R}(F_k); F_k]$ with $k \leq i$. If $R \in \mathfrak{R}(F_j)$ then $F_j - R \notin [\mathfrak{R}(F_j); F_j]$, so there is an $i < j$ with $F_j - R \in [\mathfrak{R}(F_i); F_i]$; then $F_j - R \subseteq F_i$ and hence $F_j - F_i \subseteq R$. As was just observed, $F_j - F_i$ must contain an element of $\mathfrak{R}(F_j)$; this element must be R itself, for the elements of $\mathfrak{R}(F_j)$ are pairwise disjoint. Hence $F_j - F_i = R$.

This shows that $\mathfrak{R}(F_j)$ is the set of inclusion-minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$; the elements of $\mathfrak{R}(F_j)$ are nonempty and pairwise disjoint by definition.

Conversely, suppose that for every $j > 1$ the inclusion-minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ are nonempty and pairwise disjoint. Let $\mathfrak{R}(F_1) = \emptyset$ and for $j > 1$ let $\mathfrak{R}(F_j)$ be the set of inclusion-minimal relative complements $F_j - F_i$ with $i < j$. If S is a subset of F_j which is not an element of $[\mathfrak{R}(F_j); F_j]$ then S must be disjoint from $F_j - F_i$ for some $i < j$, and hence $S \subseteq F_i$. If S is not an element of $[\mathfrak{R}(F_i); F_i]$ then the same argument shows that S must be a subset of an F_h with $h < i$; continuing in this vein we must ultimately find a $g < j$ with $S \in [\mathfrak{R}(F_g); F_g]$. Also, if $i < j$ then there is an $R \in \mathfrak{R}(F_j)$ with $R \subseteq F_j - F_i$, so no subset of F_i intersects R and consequently $[\mathfrak{R}(F_i); F_i]$ and $[\mathfrak{R}(F_j); F_j]$ are disjoint. Thus the MVI intervals $[\mathfrak{R}(F_j); F_j]$ partition $\mathfrak{F}$. ■

Corollary 2.2. *A list $F_1, ..., F_t$ is an $\mathcal{S}$-partition if and only if for every $j > 1$, the inclusion-minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ are all singletons. Moreover, if $j > 1$ then $\mathfrak{R}(F_j)$ is precisely the union of these singletons.*

Proof. As noted in the introduction, a Boolean interval is a special type of MVI interval. It follows that an $\mathcal{S}$-partition is a special type of MVI $\mathcal{S}$-partition, one in which every $\mathfrak{R}(F_j)$ with $j > 1$ consists solely of singletons. ■

The Rearrangement Lemma for MVI $\mathcal{S}$-partitions is proven in two stages.

The Transposition Lemma for MVI $\mathcal{S}$-Partitions. *Suppose $F_1, ..., F_t$ is an MVI $\mathcal{S}$-partition and $a < t$. Then $F_1, ..., F_{a-1}$, F_{a+1}, F_a, F_{a+2}, ..., F_t is not an MVI $\mathcal{S}$-partition with the same restriction function $\mathfrak{R}$ if and only if $F_{a+1} \cap F_a \in [\mathfrak{R}(F_a); F_a]$. Moreover, if this is the case then $F_{a+1} - F_a \in \mathfrak{R}(F_{a+1})$.*

Proof. If $F_1, ..., F_{a-1}, F_{a+1}, F_a, F_{a+2}, ..., F_t$ is not an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$, there must be an $S \subseteq F_{a+1}$ which appears in $[\mathfrak{R}(F_a); F_a]$. S cannot also appear in $[\mathfrak{R}(F_{a+1}); F_{a+1}]$, for the intervals are disjoint; hence there must be an $R \in \mathfrak{R}(F_{a+1})$ such that $R \cap S = \emptyset$.

Consider $F_{a+1} - R$. It is contained in F_{a+1} and does not appear in $[\mathfrak{R}(F_{a+1}); F_{a+1}]$, so $F_{a+1} - R \in [\mathfrak{R}(F_i); F_i]$ for some $i \leq a$. If $i < a$ then $S \subseteq F_{a+1} - R \subseteq F_i$ and hence S appears in $[\mathfrak{R}(F_j); F_j]$ for some $j \leq i$, violating the disjointness of $[\mathfrak{R}(F_j); F_j]$ and $[\mathfrak{R}(F_a); F_a]$. It follows that $i = a$, and hence that $F_{a+1} - R \subseteq F_a$; equivalently, $F_{a+1} - F_a \subseteq R$. By

Proposition 2.1 this implies that $F_{a+1} - F_a = R$ and consequently that $F_{a+1} \cap F_a = F_{a+1} - R \in [\mathfrak{R}(F_a); F_a]$.

Conversely, if $F_{a+1} \cap F_a \in [\mathfrak{R}(F_a); F_a]$ then clearly $F_1, ..., F_{a-1}, F_{a+1}, F_a, F_{a+2}, ..., F_t$ is not an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$, because $F_{a+1} - F_a$ is a subset of F_{a+1} which appears in an MVI interval listed after $[\mathfrak{R}(F_{a+1}); F_{a+1}]$. ■

The Rearrangement Lemma for MVI $\mathcal{S}$-Partitions. *An MVI $\mathcal{S}$-partition $F_1, ..., F_t$ may be rearranged as an MVI $\mathcal{S}$-partition $F'_1, ..., F'_t$ with the same restriction function, such that for each a either $|F'_a| \geq |F'_{a+1}|$ or $F'_a \cap F'_{a+1} \in [\mathfrak{R}(F'_a); F'_a]$.*

Proof. Let $b = b(F_1, ..., F_t)$ be the least a with $|F_a| < |F_{a+1}|$ and $F_a \cap F_{a+1} \notin [\mathfrak{R}(F_a); F_a]$. If there is no such a then let $b(F_1, ..., F_t) = t$ and observe that the proposition is satisfied without any rearrangement.

Suppose $b = t - 1$. The Transposition Lemma implies that $F_1, ..., F_{t-2}, F_t, F_{t-1}$ is an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$. If $|F_{t-2}| \geq |F_t|$ or $F_t \cap F_{t-2} \in [\mathfrak{R}(F_{t-2}); F_{t-2}]$ then this rearrangement satisfies the proposition, because $|F_{t-1}| < |F_t|$. Otherwise the Transposition Lemma implies that $F_1, ..., F_{t-3}, F_t, F_{t-2}, F_{t-1}$ is an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$. Continuing in this vein, if we do not find a rearrangement that satisfies the proposition we will eventually conclude that $F_1, F_t, F_2, ..., F_{t-1}$ is an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$. This rearrangement satisfies the proposition because $F_1 \cap F_t \in [\mathfrak{R}(F_1); F_1] = [\emptyset, F_1]$.

The proof proceeds by induction on $t-b > 1$. There must be a $k < b$ such that $F_1, ..., F_k, F_{b+1}, F_{k+1}, ..., F_b, F_{b+2}, ..., F_t$ is an MVI $\mathcal{S}$-partition with the same restrictions as $F_1, ..., F_t$ and either $|F_k| \geq |F_{b+1}|$ or $F_{b+1} \cap F_k \in [\mathfrak{R}(F_k); F_k]$, for 1 will be such a k if there is no greater such k. Then $b(F_1, ..., F_k, F_{b+1}, F_{k+1}, ..., F_b, F_{b+2}, ..., F_t) > b$ and the inductive hypothesis applies. ■

$\mathcal{S}$-partitions are simply MVI $\mathcal{S}$-partitions for which the restrictions $\mathfrak{R}(F_j)$, $j > 1$, contain only singletons. As the Transposition and Rearrangement Lemmas for MVI $\mathcal{S}$-partitions are concerned with rearrangements which do not alter the restrictions, they apply directly to $\mathcal{S}$-partitions. We deduce the following results regarding rearrangements of $\mathcal{S}$-partitions.

The Transposition Lemma for $\mathcal{S}$-Partitions. *Suppose $F_1, ..., F_t$ is an $\mathcal{S}$-partition and $a < t$. Then $F_1, ..., F_{a-1}, F_{a+1}, F_a, F_{a+2}, ..., F_t$ is not an $\mathcal{S}$-partition with the same restriction function $\mathfrak{R}$ if and only if $F_{a+1} \cap F_a \in [\mathfrak{R}(F_a), F_a]$. Moreover if this is the case then $F_{a+1} - F_a \subseteq \mathfrak{R}(F_{a+1})$ and $|F_{a+1} - F_a| = 1$.*

The Rearrangement Lemma for $\mathcal{S}$-Partitions. *An $\mathcal{S}$-partition $F_1, ..., F_t$ may be rearranged as an $\mathcal{S}$-partition $F'_1, ..., F'_t$ with the same restriction function, such that for each a either $|F'_a| \geq |F'_{a+1}|$ or there is an $x \in F'_{a+1}$ with $F'_a = F'_{a+1} - \{x\}$.*

Proof. The Rearrangement Lemma for MVI $\mathcal{S}$-partitions provides a rearrangement $F'_1, ..., F'_t$ with the same restriction function, such that for each a either $|F'_a| \geq |F'_{a+1}|$ or $F'_a \cap F'_{a+1} \in [\mathfrak{R}(F'_a), F'_a]$. If $|F'_a| < |F'_{a+1}|$, then $F'_a \cap F'_{a+1} \in [\mathfrak{R}(F'_a), F'_a]$; the Transposition Lemma for $\mathcal{S}$-partitions tells us that $F'_{a+1} - F'_a \subseteq \mathfrak{R}(F'_{a+1})$ and $|F'_{a+1} - F'_a| = 1$. Together, $|F'_a| < |F'_{a+1}|$ and $|F'_{a+1} - F'_a| = 1$ imply that there is an $x \in F'_{a+1}$ with $F'_a = F'_{a+1} - \{x\}$. ■

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