Generalized dice: many questions and a few answers

Lorenzo Traldi
Department of Mathematics
Lafayette College
Easton, Pennsylvania 18042, U.S.A.
www.lafayette.edu/~traldil/

Abstract

A *generalized die* is a list $(x_1, ..., x_n)$ of integers. A die X is *stronger* than a die Y if there are fewer pairs (i, j) with $x_i < y_j$ than pairs (i, j) with $x_i > y_j$; if neither of X, Y is stronger than the other then X and Y are *tied.* We survey known results and open problems regarding these notions.

1. Introduction

In the simplest standard dice game, two players roll identical dice; if the results are different then the winner is the player whose die has rolled the higher value. Presuming the dice are truly identical, the winner is determined by simple chance – or "luck" – and in the long run, the two players will tend to have equal numbers of wins and losses. This paper is about generalized dice games, in which the two players roll non-identical dice. In the long run two generalized dice need not tend to have equal numbers of wins and losses, even if their rolls have the same expected value.

We represent a generalized die by a list $X = (x_1, ..., x_n)$ of integers with $x_1 \leq x_2 \leq ... \leq x_n$; $x_1, ..., x_n$ are the *labels* of X. If a, b, n, and s are integers with $a \leq b$ and $n > 0$ then $D(n, a, b, s)$ is the *dice family* containing all generalized dice $(x_1, ..., x_n)$ with $a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. If $X, Y \in D(n, a, b, s)$ then X is *stronger* than Y if $|\{(i, j) \mid x_i > y_j\}| > |\{(i, j) \mid x_i < y_j\}|$; if neither of X, Y is stronger than the other then X and Y are *tied.* A die is *balanced* in $D(n, a, b, s)$ if it ties every element of $D(n, a, b, s)$.

Generalized dice games have many interesting properties, some of which are counterintuitive or even "paradoxical."

- The *stronger* relation is not transitive in general [1, 3, 4].
- If $n \leq 2$ then every element of $D(n, a, b, s)$ is balanced, but if $n \geq 3$ then balanced dice are rare [5].
- More generally, ties are less "typical" than one might expect [2].
- The structure of $D(n, a, b, s)$ varies significantly with n, a, b, and s.

Generalized dice games seem very natural, and one might well imagine that sometime during the thousands of years people have played with dice, someone might have investigated them. Instead they have been studied very little, and much remains to be discovered about them.

2. Characteristic vectors and balanced dice

It can happen that a or b is not actually a label of any element of $D(n, a, b, s)$; for instance no element of $D(5, 1, 8, 10)$ has 8 as a label. Let $p = p(n, a, b, s) = \min\{x_1 \mid (x_1, ..., x_n) \in D(n, a, b, s)\}$ and $q = q(n, a, b, s) = \max\{x_n \mid (x_1, ..., x_n) \in D(n, a, b, s)\}$; then $D(n, a, b, s) = D(n, p, q, s)$ and both p and q appear

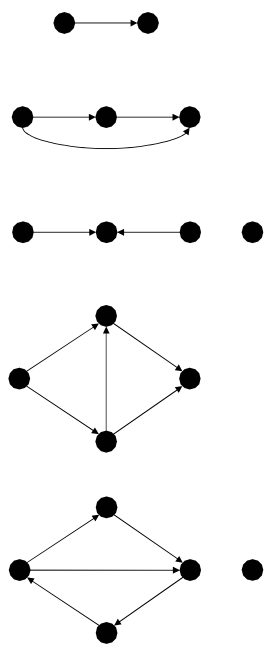

Figure 3.1: The directed dice graphs $\Delta(4,1,4,s)$, $6 \le s \le 10$

on some element of $D(n,a,b,s)$. A die $X = (x_1, ..., x_n) \in D(n,a,b,s)$ is determined by its *characteristic vector* $v^X = (v_p^X, ..., v_q^X)$, with $v_i^X = |\{j \mid x_j = i\}|$. If $n \ge 3$ then X is balanced if and only if v^X is of the form $(v, w, v, w, ...)$ [5]; for instance $(2,3,4)$ is balanced in $D(3,2,4,9)$, where its characteristic vector is $(1,1,1)$, but not in $D(3,1,4,9)$, where its characteristic vector is $(0,1,1,1)$. It follows that balanced dice are rather rare; for instance there are only three balanced dice among the 458 elements of the dice families $D(6,1,6,s)$ with $8 \le s \le 34$ [5].

3. Transitivity and non-transitivity

Generalized dice first attracted attention because of the non-transitivity of *stronger* [1, 3, 4]; for example, consider $(3,3,4)$, $(2,2,6)$, $(1,4,5) \in D(3,1,5,10)$. The relation *tied* is also non-transitive in general; for example $(1,4,5)$, $(2,3,5)$, and $(2,4,4)$. There is a *directed dice graph* $\Delta(n,a,b,s)$ naturally associated with $D(n,a,b,s)$; it has a vertex for each $X \in D(n,a,b,s)$, and an edge directed from X to Y if X is stronger than Y. Non-transitivity of *stronger* can create directed cycles in $\Delta(n,a,b,s)$.

A special feature of the dice families with $q - p \le 2$ is that they have a numerical measure of strength: there is a function $str : D(n,a,b,s) \to \mathbb{Q}$ such that X is stronger than Y if and only if $str(X) > str(Y)$. Moreover, str is either injective or identically 0 [6, 7]. Consequently, if $q - p \le 2$ then *stronger* is either linear or trivial; in any case it is transitive, and $\Delta(n,a,b,s)$ is acyclic. The interested reader might like to try to find the definition of str as an exercise – it is not the mean label value.

If $q - p > 2$ then just about every question one might ask about the structure of $\Delta(n,a,b,s)$ is open; we do not know of any results more substantial than the observation that these digraphs contain more directed cycles if s/n is close to $(p+q)/2$ rather than p or q. A very simple illustration of this observation is given in Figure 3.1, where the five directed dice graphs $\Delta(4,1,4,s)$, $6 \le s \le 10$ are pictured. Each is more complicated than the one before: $\Delta(4,1,4,6)$ and $\Delta(4,1,4,7)$ are linear, $\Delta(4,1,4,8)$ is transitive but not linear, $\Delta(4,1,4,9)$ is acyclic but not transitive, and $\Delta(4,1,4,10)$ has directed cycles.

- Which $\Delta(n, a, b, s)$ are linear, or transitive, or acyclic?
- Which $\Delta(n, a, b, s)$ are covered by cycles?
- Which $\Delta(n, a, b, s)$ are covered by 3-cycles?
- Which $\Delta(n, a, b, s)$ contain vertices whose in- and out-degrees are nonzero and equal?
- Which $\Delta(n, a, b, s)$ contain sources or sinks?

Proposition 3.1 tells us that some sources and sinks in directed dice graphs are associated with balanced dice; the sources have characteristic vectors $(v, w, ..., v \text{ or } w, 0, ..., 0)$ and the sinks have characteristic vectors $(0, ..., 0, v, w, ..., v \text{ or } w)$. But these are not the only sources and sinks; for instance $\Delta(6, 1, 6, 13)$ has the sink $(1, 1, 1, 1, 3, 6)$ and the source $(1, 1, 2, 3, 3, 3)$.

Proposition 3.1. *If X is balanced in $D(n, a, b, s)$ then X is not weaker than any element of $D(n, p, \beta, s)$ for any $\beta \geq q$, and X is not stronger than any element of $D(n, \alpha, q, s)$ for any $\alpha \leq p$.*

Proof. As X is balanced, $v^X = (v, w, v, w, ...)$ [5]; equivalently $v_i^X + v_{i+1}^X = v_j^X + v_{j+1}^X$ for $i, j \in \{p, ..., q-1\}$. Suppose $Y = (y_1, ..., y_n) \in D(n, p, \beta, s)$, where $\beta \geq q$. If $y_1 = q$ then $s \geq nq$, so $s = nq = np$ and consequently $Y = X$ and X ties Y; hence we may presume $y_1 < q$. If $y_n \leq b$ then $Y \in D(n, a, b, s)$ so X ties Y. If $y_n > b$ then let $Y' = (y_1 + 1, y_2, ..., y_{n-1}, y_n - 1)$; the labels of Y' may not appear in the indicated order, but this technicality does not affect the argument. We may presume inductively that X is not weaker than Y'. If we compare the performance of Y against X with that of Y' against X, we see that the label $y_1 + 1$ of Y' has $v_{y_1}^X$ more winning rolls and $v_{y_1+1}^X$ fewer losing rolls than the corresponding label y_1 of Y, and the label $y_n - 1$ of Y' has $v_{y_n-1}^X$ fewer winning rolls and $v_{y_n}^X = 0$ more losing rolls than the corresponding label y_n of Y. As $y_1 < q$, $v_{y_1}^X + v_{y_1+1}^X = v_{q-1}^X + v_q^X \geq v_{y_n-1}^X + v_{y_n}^X = v_{y_n-1}^X$, so Y cannot do better against X than Y' does.

The argument that X is not stronger than any element of $D(n, \alpha, q, s)$ is essentially the same. ■

4. Ties

There are many interesting questions suggested by the observation that in general, there are fewer ties among generalized dice than one might expect.

- Are there any dice families $D(n, a, b, s)$ in which most pairs of distinct, non-balanced dice are tied?
- What can be said about asymptotic behavior of the relative proportion of ties in $D(n, a, b, s)$?

There are many limits to which "asymptotic" might refer. For instance one might consider $D(n, a-x, b+x, s)$ as $x \to \infty$, $D(n, a, b, s)$ as $n \to \infty$, $D(n, a-n, b+n, s)$ as $n \to \infty$, etc.

- How common are non-balanced dice most of whose results are ties?

Such dice seem to be relatively rare, but one dice family may contain several. For instance, among the 151 elements of $D(6, 1, 9, 30)$ there are five which tie more than 75 of their companions. None of these five has more than 81 ties, though; this suggests a more precise question.

- For various values of $\varepsilon > 0$, how common are non-balanced dice whose proportions of ties lie in the interval $(.5 + \varepsilon, 1)$?
- How are ties (or non-ties) distributed? Do they tend to "clump," or are they "spread out"?

A natural way to think about "clumping" is to consider the structure of the *undirected dice graph* $G(n, a, b, s)$ obtained from $\Delta(n, a, b, s)$ by ignoring edge-directions.

- What can be said about the vertex- or edge-connectivity of $G(n, a, b, s)$?

A recent result implies that up to isomorphism, $G(3,1,9,15)$ is the only dice graph whose diameter is more than 2.

The Tied Dice Theorem. [2] *Suppose $X \neq Y \in D(n,a,b,s)$ are tied, non-balanced dice. Unless X and Y are the dice $(a, a+4, a+8)$ and $(a+2, a+4, a+6)$ in $D(3, a, a+8, 3a+12)$, there is a $Z \in D(n,a,b,s)$ which ties neither X nor Y.*

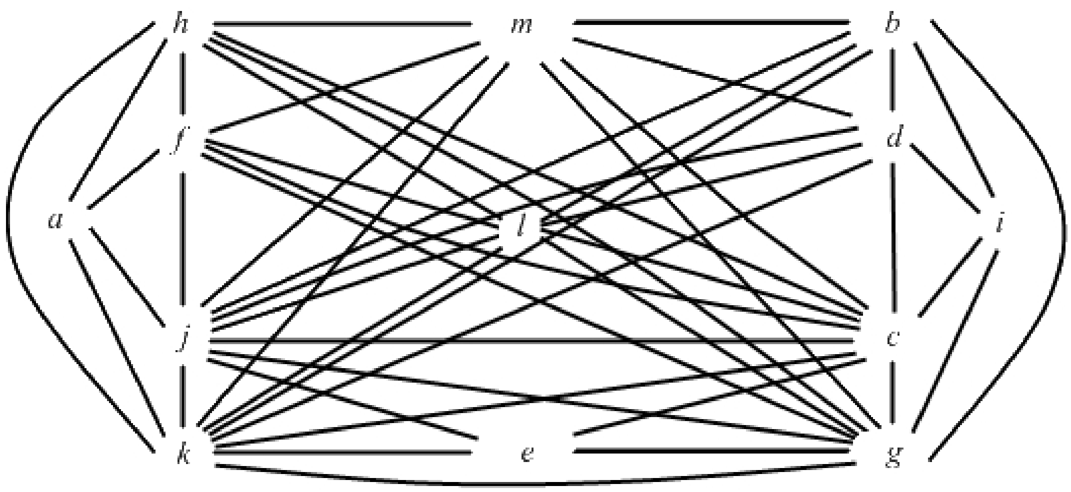

Figure 4.1: Up to isomorphism, $G(3,1,9,15)$ is the only dice graph of diameter > 2. (Vertices are labeled lexicographically – a represents $(1,5,9)$, b represents $(1,6,8)$, etc.)

In conclusion, we hope the reader will agree that generalized dice are interesting objects that merit more attention than they have received.

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