Available online at www.sciencedirect.com
SCIENCE DIRECT®
Discrete Mathematics 290 (2005) 291–299
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Parallel connections and coloured Tutte polynomials

Lorenzo Traldi

Department of Mathematics, Lafayette College, Easton, PA 18042, USA

Received 24 April 2002; received in revised form 16 July 2004; accepted 4 August 2004
Available online 12 January 2005

Abstract

We give formulas for the Tutte polynomials of parallel and series connections of weighted matroids, generalizing and unifying formulas of Oxley and Welsh (Discrete Math. 109 (1992) 185), Bollobás and Riordan (Comb. Probab. Comput. 8 (1999) 45) and Woodall (Discrete Math. 247 (2002) 201).

Keywords: Parallel; Series; Coloured Tutte polynomial; Tension polynomial; Flow polynomial

1. Introduction

The *Tutte polynomial* is one of the most-studied invariants of graphs and matroids. It may be defined in several equivalent ways, including a *deletion–contraction recursion*: if M is a matroid on a finite set $E = E(M)$ then $T(M)$ is an element of the polynomial ring $\mathbb{Z}[X, Y]$ such that $T(M) = XT(M/e)$ for any coloop e of M, $T(M) = YT(M - e)$ for any loop e of M, and $T(M) = T(M - e) + T(M/e)$ for any other $e \in E$. The empty matroid $\emptyset$ has $T(\emptyset) = 1$. (A related invariant of graphs, the *dichromatic polynomial*, follows the same recursion but has a more complicated initial condition: an edgeless graph G has dichromatic polynomial $(X - 1)^{|V(G)|}$.) The Tutte polynomial may also be defined by *basis activities*, which we will not use in this note, or the *subset expansion*

$$T(M) = \sum_{S \subseteq E} (X - 1)^{r(E) - r(S)} (Y - 1)^{|S| - r(S)}.$$

E-mail address: traldil@lafayette.edu (L. Traldi).

doi:10.1016/j.disc.2004.08.018

We refer the reader to [1,12,13,15] for detailed discussions, and to [6,14] for the theory of matroids.

The Tutte polynomial gives information about many aspects of the structure of a graph or matroid, including vertex colourings, flows, network reliability, polynomial invariants of knots, and quantities of interest in statistical mechanics. Some of these involve graphs and matroids which have been weighted in various ways; for instance, in statistical mechanics the edges of a lattice might represent hydrogen bonds between nearby water molecules, weighted to indicate the probabilities (as functions of temperature) that these bonds hold. Many authors have considered many different weighted versions of the Tutte polynomial [2,4,5,8–10,16,17]. (We use the general term *weighted* to include the *coloured*, *parametrized*, *uncoloured* and *labeled* invariants discussed in the references.) The two primary references, in which such weighted Tutte polynomials have been not only defined but characterized (under different hypotheses) are [2] and [17].

Recall that if M_1 and M_2 are matroids on sets E_1 and E_2 with $E_1 \cap E_2 = \{e_0\}$ and e_0 is neither a loop nor a coloop in either then the *parallel connection* of M_1 and M_2 is the matroid M on $E = E_1 \cup E_2$ with circuit-set $C(M) = C(M_1) \cup C(M_2) \cup \{C_1 \cup C_2 - \{e_0\} \mid e_0 \in C_1 \in C(M_1)$ and $e_0 \in C_2 \in C(M_2)\}$; $M - e_0$ is the 2-*sum* of M_1 and M_2. We use the same terminology for the corresponding constructions on graphs: the union of two graphs which share only a single non-loop, non-isthmus edge e_0 (and its end-vertices) is their parallel connection and the 2-sum of the two graphs is obtained from their parallel connection by removing e_0. In [7] Oxley and Welsh make use of a formula relating the Tutte polynomial of a 2-sum of matroids to those of its summands, derived from earlier work of Brylawski [3]. Woodall [16] gives 2-sum formulas for several weighted polynomial invariants of graphs. In [2,9,10] it is observed that a set of parallel edges in a graph G may be replaced by a single edge without changing the values of certain weighted Tutte polynomials, as long as the new edge is weighted appropriately. Our central result is a common generalization of all of these.

Despite the title of the note, we prove our result for the *uncoloured* version of the Bollobás–Riordan matroid invariant W [2], which we denote W^u. All the other weighted Tutte polynomials in the literature are multiples of evaluations of this one, so formulas appropriate to them can be obtained directly from formulas involving W^u. Before stating the result we must define W^u.

For a finite set E let $\mathbb{Z}[\{x_e, y_e, X_e, Y_e \mid e \in E\}]$ be the polynomial ring with four distinct indeterminates for each element of E, and let $I_0^u(E)$ be the ideal generated by the various elements

$$X_{e_1} y_{e_2} - y_{e_1} X_{e_2} - x_{e_1} Y_{e_2} + Y_{e_1} x_{e_2},$$

$$Y_{e_3}(x_{e_1} Y_{e_2} - Y_{e_1} x_{e_2} - x_{e_1} y_{e_2} + y_{e_1} x_{e_2})$$

and

$$X_{e_3}(x_{e_1} Y_{e_2} - Y_{e_1} x_{e_2} - x_{e_1} y_{e_2} + y_{e_1} x_{e_2})$$

with $e_1, e_2, e_3 \in E$ pairwise distinct. We denote the quotient ring $\mathbb{Z}[\{x_e, y_e, X_e, Y_e | e \in E\}]/I_0^u(E)$ by $\mathbb{Z}_E^u$. The elements of this quotient ring are cosets and the quotient ring is a module over the polynomial ring, so (for instance) $x_{e_1} Y_{e_2}$ does not denote an element of

$\mathbb{Z}_E^u$ while $x_{e_1}Y_{e_2} + I_0^u(E)$, $x_{e_1}(Y_{e_2} + I_0^u(E))$, $(x_{e_1} + I_0^u(E))Y_{e_2}$, and $(Y_{e_2} + I_0^u(E))(x_{e_1} + I_0^u(E))$ all denote the same element of $\mathbb{Z}_E^u$. To avoid unduly cluttered notation, if $E' \subseteq E$ we do not explicitly indicate the ring homomorphism $\mathbb{Z}_{E'}^u \to \mathbb{Z}_E^u$ naturally induced by inclusion.

Definition 1. Let M be a matroid on the finite set E. Then $W^u(M)$ is the element of $\mathbb{Z}_E^u$ defined by the properties $W^u(\emptyset) = 1 + I_0^u(E)$, $W^u(N) = X_e W^u(N/e)$ for a coloop e of a matroid N, $W^u(N) = Y_e W^u(N-e)$ for a loop e of a matroid N, and $W^u(N) = y_e W^u(N-e) + x_e W^u(N/e)$ for any other element e of a matroid N.

We refer to [2] for the fact that $W^u(M)$ is well-defined, i.e. that all recursive computations using Definition 1 have the same result.

Theorem 1. *Suppose M is the parallel connection of M_1 and M_2 with respect to an element e_0 which is neither a loop nor a coloop. Let $f : \mathbb{Z}_{E_1}^u \to \mathbb{Z}_E^u$ be the ring homomorphism given by*

$$f(x_e + I_0^u(E_1)) = x_e + I_0^u(E) \quad for\ e \neq e_0,$$

$$f(y_e + I_0^u(E_1)) = y_e + I_0^u(E) \quad for\ e \neq e_0,$$

$$f(X_e + I_0^u(E_1)) = X_e + I_0^u(E) \quad for\ e \neq e_0,$$

$$f(Y_e + I_0^u(E_1)) = Y_e + I_0^u(E) \quad for\ e \neq e_0,$$

$$\begin{aligned} f(x_{e_0} + I_0^u(E_1)) = {} & x_{e_0}(x_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2/e_0) \\ & + x_{e_0}y_{e_0}^2 W^u(M_2 - e_0), \end{aligned}$$

$$f(y_{e_0} + I_0^u(E_1)) = y_{e_0}^2 x_{e_0} W^u(M_2/e_0) + y_{e_0}^2(y_{e_0} - Y_{e_0})W^u(M_2 - e_0),$$

$$f(X_{e_0} + I_0^u(E_1)) = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2)$$

and

$$f(Y_{e_0} + I_0^u(E_1)) = Y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2/e_0).$$

Then

$$f(W^u(M_1)) = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M).$$

If M^* is the dual matroid of M then $W^u(M^*)$ can be obtained from $W^u(M)$ by interchanging $x_e \leftrightarrow y_e$ and $X_e \leftrightarrow Y_e$ for every e, because deletion and contraction are dual operations, and coloops and loops are dual elements. Hence Theorem 1 has a dual which describes W^u of a series connection; we leave its formulation to the reader.

A simple induction proves that $W^u(M' \bigoplus M'') = W^u(M')W^u(M'')$; see Remark 6 of [2]. This property and Theorem 1 imply the following.

Theorem 2. *Suppose M is the parallel connection of M_1 and M_2 with respect to an element e_0 which is neither a loop nor a coloop. Then for the* 2*-sum $M - e_0$ we have*

$$\begin{aligned} &y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M - e_0) \\ &\quad = y_{e_0}^2(y_{e_0} - Y_{e_0})W^u(M_2 - e_0)W^u(M_1 - e_0) + y_{e_0}^2 x_{e_0} W^u(M_2/e_0)W^u(M_1 - e_0) \\ &\qquad + x_{e_0}y_{e_0}^2 W^u(M_2 - e_0)W^u(M_1/e_0) \\ &\qquad + x_{e_0}y_{e_0}(x_{e_0} - X_{e_0})W^u(M_1/e_0)W^u(M_2/e_0). \end{aligned}$$

Perhaps the most intuitively immediate weighted version of the Tutte polynomial is the one associated to a *doubly weighted* matroid *M*, i.e. a matroid *M* on a finite set *E* which has been equipped with two functions, denoted $e \mapsto p_e$ and $e \mapsto q_e$, mapping *E* into some commutative ring with unity *R*. The Tutte polynomial of such an M is defined by modifying the subset expansion of the ordinary Tutte polynomial to take the weights into account

$$T(M) = \sum_{S \subseteq E} \left(\prod_{e \in S} p_e \right) \left(\prod_{e \in E-S} q_e \right) (X-1)^{r(E)-r(S)}(Y-1)^{|S|-r(S)}.$$

(*X* and *Y* may be indeterminates, so that $T(M)$ is an element of a polynomial ring $R[X, Y]$, or more generally *X*, *Y* and $T(M)$ may all be elements of *R*.) To appreciate the significance of $T(M)$, suppose *M* is the cycle matroid of a connected graph *G*, whose edges may fail independently of each other. Suppose also that for each $e \in E = E(G)$, p_e gives the probability that *e* successfully connects its end-vertices and $q_e = 1 - p_e$ gives the probability that *e* fails. If *X* and *Y* are distinct indeterminates then the coefficient of $(X-1)^a(Y-1)^b$ in $T(M)$ is the probability that a randomly chosen subset $S \subseteq E$ will have cardinality $b - a - 1 + |V(G)|$ and will induce a subgraph with $a + 1$ connected components.

It is a simple exercise to verify that $T(M)$ satisfies a doubly weighted deletion–contraction recursion: $T(\emptyset) = 1$, $T(M) = ((X-1)q_e + p_e)T(M/e)$ for any coloop *e* of *M*, $T(M) = ((Y-1)p_e + q_e)T(M-e)$ for any loop *e* of *M*, and $T(M) = q_eT(M-e) + p_eT(M/e)$ for any other $e \in E$. ($M - e$ and M/e inherit weights from *M* by restriction.) It follows that $T(M)$ is obtained from $W^u(M)$ by evaluation: $x_e \mapsto p_e$, $y_e \mapsto q_e$, $X_e \mapsto (X-1)q_e + p_e$ and $Y_e \mapsto (Y-1)p_e + q_e$. The image of $I_0^u(E)$ under these evaluations is 0, and consequently when we evaluate formulas involving W^u the resulting formulas involving *T* do not require coset notation. Also, it happens often that formulas involving W^u yield factorizable formulas involving *T*. Like factors in these formulas can be cancelled even if *R* is not assumed to be an integral domain, because (a) the like factors appear in the formulas involving the doubly weighted version of $T(M)$ with the natural weights in the polynomial ring $\mathbb{Z}[\{x_e, y_e | e \in E\}]$, where cancellation may be performed, and then (b) the simplified formula can be evaluated in the ring *R*, mapping each $x_e \mapsto p_e$ and each $y_e \mapsto q_e$.

In this way Theorems 1 and 2 imply the following.

Corollary 1. *Let M be a doubly weighted matroid which is the parallel connection of M_1 and M_2 with respect to an element e_0 which is neither a loop nor a coloop. Let M_1' be the* *doubly weighted matroid on* E_1 *whose underlying unweighted matroid is* M_1 *and whose weight functions* p' *and* q' *coincide with p and q* (*resp.*) *on* $E_1 - \{e_0\}$ *and have*

$$p'_{e_0} = ((1 - X)q_{e_0} + (X + Y - XY)p_{e_0}))T(M_2/e_0) + q_{e_0}T(M_2 - e_0)$$

and

$$q'_{e_0} = q_{e_0}T(M_2/e_0) + q_{e_0}(1 - Y)T(M_2 - e_0).$$

Then $T(M'_1) = (X + Y - XY)T(M)$. *Also, for the* 2-*sum* $M - e_0$ *we have*

$$\begin{aligned}&(X + Y - XY)T(M - e_0)\\ &\quad = (1 - Y)T(M_2 - e_0)T(M_1 - e_0) + T(M_2/e_0)T(M_1 - e_0)\\ &\qquad + T(M_2 - e_0)T(M_1/e_0) + (1 - X)T(M_1/e_0)T(M_2/e_0).\end{aligned}$$

The 2-sum formula of Corollary 1 is exactly the same as the formula for the unweighted Tutte polynomial of a 2-sum used in [7], generalized to doubly weighted matroids. It follows, by the way, that the analysis in [7] of the complexity of unweighted Tutte polynomial calculations for certain classes of matroids generalizes directly to an analysis of the complexity of weighted Tutte polynomial calculations for those classes of matroids.

In Section 3 we show that Corollary 1 implies the 2-sum formulas of Woodall [16].

2. Proof of Theorem 1

Theorem 1 is proven in two parts. First we must prove that the function f is well-defined, i.e. if $F : \mathbb{Z}[\{x_e, y_e, X_e, Y_e | e \in E_1\}] \to \mathbb{Z}_E^u$ is the ring homomorphism given by the formulas of Theorem 1 ($F(x_e) = x_e + I_0^u(E)$ for $e \neq e_0$, $F(x_{e_0}) = x_{e_0}(x_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2/e_0) + x_{e_0}y_{e_0}^2 W^u(M_2 - e_0)$, and so on) then whenever $e_1, e_2, e_3 \in E_1$ are pairwise distinct

$$\begin{aligned}&F(X_{e_1}y_{e_2} - y_{e_1}X_{e_2} - x_{e_1}Y_{e_2} + Y_{e_1}x_{e_2}) = 0,\\ &F(Y_{e_3}(x_{e_1}Y_{e_2} - Y_{e_1}x_{e_2} - x_{e_1}y_{e_2} + y_{e_1}x_{e_2})) = 0\end{aligned}$$

and

$$F(X_{e_3}(x_{e_1}Y_{e_2} - Y_{e_1}x_{e_2} - x_{e_1}y_{e_2} + y_{e_1}x_{e_2})) = 0. \tag{2.1}$$

If $e_0 \notin \{e_1, e_2, e_3\}$ then (2.1) is obviously true, as the listed values of F are among the defining generators of $I_0^u(E)$. Moreover, if $e_0 = e_3$ then (2.1) follows readily from the definition of $I_0^u(E)$. The first value of F listed in (2.1) is one of the defining generators of $I_0^u(E)$. Also, $F(X_{e_0})$ and $F(Y_{e_0})$ are both elements of the ideal of $\mathbb{Z}_E^u$ generated by X_{e_0} and Y_{e_0}, so the second and third equalities of (2.1) follow directly from the formulas of the second and third types of generators of $I_0^u(E)$ corresponding to e_1, e_2, e_3.

The indices e_1 and e_2 appear symmetrically in (2.1), so it suffices now to consider the possibility that $e_0 = e_1$. The reader can easily supply the necessary computations by using the definition of F, replacing $W^u(M_2)$ with $x_{e_0}W^u(M_2/e_0) + y_{e_0}W^u(M_2 - e_0)$, collecting terms and using the definition of $I_0^u(E)$ to eliminate appearances of X_{e_2} and Y_{e_2}. In using the definition of $I_0^u(E)$ it is useful to note that there is an $e_3' \in E_2 - \{e_0\}$ and that $W^u(M_2/e_0)$ and $W^u(M_2 - e_0)$ are both elements of the ideal of $\mathbb{Z}_E^u$ generated by $X_{e_3'}$ and $Y_{e_3'}$, because there are recursive computations of $W^u(M_2/e_0)$ and $W^u(M_2 - e_0)$ in which e_3' is the last element of $E_2 - \{e_0\}$ to be eliminated; consequently,

$$\begin{aligned}&W^u(M_2/e_0) \cdot (x_{e_0}Y_{e_2} - Y_{e_0}x_{e_2} - x_{e_0}y_{e_2} + y_{e_0}x_{e_2})\\ &\quad = W^u(M_2 - e_0) \cdot (x_{e_0}Y_{e_2} - Y_{e_0}x_{e_2} - x_{e_0}y_{e_2} + y_{e_0}x_{e_2}) = 0.\end{aligned}$$

We complete the proof of Theorem 1 by verifying $f(W^u(M_1)) = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M)$ by induction on $|E_1| \geqslant 2$.

Suppose $|E_1| = 2$, and let $E_1 = \{e_0, e_1\}$. Then e_0 and e_1 are parallel (and in series) in M_1, and hence

$$\begin{aligned}f(W^u(M_1)) &= f(x_{e_1}Y_{e_0} + y_{e_1}X_{e_0} + I_0^u(E))\\ &= (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})(x_{e_1}W^u(M_2) + y_{e_1}Y_{e_0}W^u(M_2/e_0)).\end{aligned}$$

A routine calculation using the deletion–contraction property of W^u shows that $x_{e_1}W^u(M_2)+ y_{e_1}Y_{e_0}W^u(M_2/e_0) = W^u(M)$.

Suppose now that $|E_1| > 2$. If there is an $e_1 \in E_1$ such that Theorem 1 holds for both M/e_1 and $M - e_1$, then the validity of Theorem 1 for M follows using the deletion–contraction formula.

If there is an $e_1 \in E_1$ which is neither parallel to nor in series with e_0 then Theorem 1 may be inductively assumed to hold for both M/e_1 and $M - e_1$.

If there is an $e_1 \in E_1$ which is parallel to e_0 then we may assume inductively that Theorem 1 holds for $M - e_1$. M/e_1 is the direct sum of M_2/e_0, a loop e_0 and $(M_1/e_1) - e_0$. It follows that

$$\begin{aligned}&(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M/e_1)\\ &\quad = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2/e_0)Y_{e_0}W^u((M_1/e_1) - e_0)\\ &\quad = f(Y_{e_0}W^u((M_1/e_1) - e_0)) = f(W^u(M_1/e_1))\end{aligned}$$

so Theorem 1 holds for M/e_1.

Finally, suppose e_0 is in series with every $e_1 \in E_1$, i.e. M_1 is a circuit. Choose a particular $e_1 \neq e_0 \in E_1$. We may assume inductively that Theorem 1 holds for M/e_1. $M - e_1$ is the direct sum of M_2 and $M_1 - e_1 - e_0$, so $W^u(M - e_1) = W^u(M_1 - e_1 - e_0)W^u(M_2)$. $M_1 - e_1$ is the direct sum of $\{e_0\}$ and $M_1 - e_1 - e_0$, so

$$\begin{aligned}f(W^u(M_1 - e_1)) &= f(W^u(M_1 - e_1 - e_0)X_{e_0})\\ &= W^u(M_1 - e_1 - e_0)(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M_2)\\ &= (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})W^u(M - e_1)\end{aligned}$$

and we see that Theorem 1 holds for $M - e_1$. $\square$

By the way, the reader who has worked through the proof will have no trouble showing that Theorem 1 also applies to the *coloured* Bollobás–Riordan invariant W. One need only verify that (2.1) also holds when e_1, e_2, e_3 are not pairwise distinct, working modulo the ideal $I_0(E)$ which has generators like those of $I_0^u(E)$, except that e_1, e_2, e_3 need not be pairwise distinct.

Theorem 2 follows readily from Theorem 1:

$$\begin{aligned}
&y_{e_0}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})W^u(M-e_0)\\
&\quad = f(W^u(M_1)) - x_{e_0}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})W^u(M_1/e_0)W^u(M_2/e_0)\\
&\quad = f(y_{e_0})W^u(M_1-e_0)+f(x_{e_0})W^u(M_1/e_0)\\
&\qquad - x_{e_0}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})W^u(M_1/e_0)W^u(M_2/e_0)\\
&\quad = (y_{e_0}^2x_{e_0}W^u(M_2/e_0)+y_{e_0}^2(y_{e_0}-Y_{e_0})W^u(M_2-e_0))W^u(M_1-e_0)\\
&\qquad + x_{e_0}((x_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})W^u(M_2/e_0)\\
&\qquad + y_{e_0}^2W^u(M_2-e_0))W^u(M_1/e_0)\\
&\qquad - x_{e_0}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})W^u(M_1/e_0)W^u(M_2/e_0).
\end{aligned}$$

3. The tension and flow polynomials

In this section we show that Corollary 1 implies the crucial formulas (3.31) and (3.32) of [16].

If M is a matroid on E then its *tension polynomial* is

$$N(M)=\sum_{S\subseteq E} q^{r(E)-r(S)}\left(\prod_{e\in S}(\beta_e-1)\right),$$

where q is an indeterminate and $\boldsymbol{\beta}=(\beta_e)$ is a vector of indeterminates indexed by E. (We use $\boldsymbol{\beta}$ rather than the $\mathbf{y}$ of [16] to avoid confusion.) Formulas (2.3) and (3.7) of [16] imply that this is a direct generalization of the definition given there: if G is a graph with no loops or coloops and M is its circuit matroid, then the tension polynomial $N(G,q,\boldsymbol{\beta})$ defined in [16] coincides with $N(M)$. If we consider M as a doubly weighted matroid with the natural weights in the polynomial ring $\mathbb{Z}[\{X,Y\}\cup\{x_e,y_e\}]$, then clearly $N(M)$ is obtained from $T(M)$ by evaluating every $x_e\mapsto\beta_e-1$, every $y_e\mapsto 1$, $X\mapsto q+1$ and $Y\mapsto 2$. (By the way, this implies that the tension polynomial is closely related to the sheaf polynomial [8,11]: $(-1)^{n-1}N(G,q,\boldsymbol{\beta})$ is $Sh(G)$ after the changes of variables $q=1-\omega$ and $\beta_e=a$.) Observe that if e is neither a loop nor a coloop of M then $N(M)=N(M-e)+(\beta_e-1)N(M/e)$.

We proceed to derive formula (3.31) of [16] from Corollary 1. Suppose M_1 and M_2 are matroids which share an element e_0 which is neither a loop nor a coloop in either, and M is their parallel connection. The deletion–contraction property of N implies that $N(M_1)=N(M_1-e_0)+(\beta_{e_0}-1)N(M_1/e_0)$. Consequently if we evaluate

$$\beta_{e_0}\mapsto\frac{(1-q)N(M_2/e_0)}{N(M_2/e_0)-N(M_2-e_0)}$$

then the image of $N(M_1)$ is

$$N(M_1-e_0)+\left(\frac{-qN(M_2/e_0)+N(M_2-e_0)}{N(M_2/e_0)-N(M_2-e_0)}\right)N(M_1/e_0).$$

Corollary 1 directly implies that

$$(1-q)N(M-e_0) = N(M_1-e_0)(N(M_2/e_0)-N(M_2-e_0)) \\ + N(M_1/e_0)(N(M_2-e_0)-qN(M_2/e_0))$$

and hence the image of $(N(M_2/e_0)-N(M_2-e_0))N(M_1)$ is $(1-q)N(M-e_0)$. Aside from changes of notation (β_{e_0} for $y(G,e')$, etc.), this is precisely the matroid generalization of formula (3.31) of [16].

We turn now to formula (3.32) of [16]. If M is a matroid on E then its *flow polynomial* is

$$F(M) = \sum_{S\subseteq E} q^{|S|-r(S)} \left(\prod_{e\notin S}(\alpha_e-1)\right),$$

where q is an indeterminate and $\boldsymbol{\alpha}=(\alpha_e)$ is a vector of indeterminates indexed by E. Formulas (2.4) and (3.8) of [16] imply that if G is a graph with no loops or coloops and M is its circuit matroid then $F(M)$ coincides with the flow polynomial $F(G,q,\boldsymbol{\alpha})$ as defined in [16]. Observe that if we consider M as a doubly weighted matroid with weights in the polynomial ring $\mathbb{Z}[\{X,Y\}\cup\{x_e,y_e\}]$, then $F(M)$ is obtained from $T(M)$ by evaluating every $x_e \mapsto 1$, every $y_e \mapsto \alpha_e - 1$, $X \mapsto 2$ and $Y \mapsto q+1$. This implies also that $F(G,q,\boldsymbol{\alpha})$ is equal to the Read–Whitehead chain polynomial [8,11] after the changes of variables $q=1-\omega$, $\alpha_e = a$.

Theorem 3.1 of [16] relates the tension and flow polynomials

$$F(M) = \left(\prod_{e\in E}(\alpha_e-1)\right) q^{-r(E)} N(M).$$

It may be deduced from the above descriptions of $F(M)$ and $N(M)$ as sums indexed by the subsets of E, using the fact that $(\alpha_e-1)(\beta_e-1)\equiv q$. Formula (3.32) of [16] follows from formula (3.31) and this relationship between $F(M)$ and $N(M)$.

Theorem 3.1 of [16] is an interesting duality connecting the tension and flow polynomials, but it is not the only type of duality that connects them. Recall that the *dual* of M is the matroid M^* on E with rank function $r^*(S) = |S| - r(E) + r(E-S)$. A direct calculation shows that $N(M^*) = F(M)$ except for the appearance of $\boldsymbol{\beta}$ in $N(M^*)$ and $\boldsymbol{\alpha}$ in $F(M)$. In [16] the tension polynomial of a graph G is defined in terms of *q-tensions*, which are related to the circuits of G, and the flow polynomial is defined in terms of *q-flows*, which are related to the cocircuits of G. The equation $N(M^*) = F(M)$ reflects the duality between circuits and cocircuits.

Acknowledgements

We are grateful to the anonymous referees for their advice on earlier versions of the note.

References

[1] B. Bollobás, Modern Graph Theory, Springer, New York, 1998.
[2] B. Bollobás, O. Riordan, A Tutte polynomial for coloured graphs, Comb. Probab. Comput. 8 (1999) 45–93.
[3] T. Brylawski, A combinatorial model for series-parallel networks, Trans. Amer. Math. Soc. 154 (1971) 1–22.
[4] J.W. Essam, Graph theory and statistical physics, Discrete Math. 1 (1971) 83–112.
[5] C.M. Fortuin, P.W. Kasteleyn, On the random-cluster model I, Introduction and relation to other models, Physica 57 (1972) 536–564.
[6] J.G. Oxley, Matroid Theory, Oxford University Press, Oxford, 1992.
[7] J.G. Oxley, D.J.A. Welsh, Tutte polynomials computable in polynomial time, Discrete Math. 109 (1992) 185–192.
[8] R.C. Read, E.G. Whitehead Jr., Chromatic polynomials of homomorphism classes of graphs, Discrete Math. 204 (1999) 337–356.
[9] L. Traldi, A dichromatic polynomial for weighted graphs and link polynomials, Proc. Amer. Math. Soc. 106 (1989) 279–286.
[10] L. Traldi, Series and parallel reductions for the Tutte polynomial, Discrete Math. 220 (2000) 291–297.
[11] L. Traldi, Chain polynomials and Tutte polynomials, Discrete Math. 248 (2002) 279–282.
[12] W.T. Tutte, Graph Theory, Cambridge University Press, Cambridge, 1984.
[13] D.J.A. Welsh, Complexity: Knots, Colourings and Counting, Cambridge University Press, Cambridge, 1995.
[14] N. White (Ed.), Theory of Matroids, Cambridge University Press, Cambridge, 1986.
[15] N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, 1992.
[16] D.R. Woodall, Tutte polynomial expansions for 2-separable graphs, Discrete Math. 247 (2002) 201–213.
[17] T. Zaslavsky, Strong Tutte functions of matroids and graphs, Trans. Amer. Math. Soc. 334 (1992) 317–347.