Several definitions of matroids *

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Abstract

We discuss several equivalent definitions of matroids, motivated by the single forbidden minor of matroid basis clutters.

1. A definition

A distinctive aspect of the elementary theory of matroids is that they can be described in so many equivalent ways; for instance West gives thirteen different axiomatizations in Chapter 8 of [8]. In this expository note we present a unified treatment of several of these descriptions, and some new ones. We hope the note will be both accessible to novices and interesting to cognoscenti.

Let E be a finite set. If E were a spanning set of a vector space then some of its subsets would be bases of the vector space, and these bases would be related to each other in various ways dictated by the theorems of linear algebra. In *matroid theory* we generalize these relationships among bases, and study families of subsets of finite sets which satisfy the generalized relationships. (It is also interesting to generalize properties of other types of sets which are important in linear algebra — spanning sets, linearly dependent sets, etc. — but we will focus on bases for now.)

The first step in this generalization process is to answer a question. Suppose E is a spanning set of a vector space $\mathcal{V}$. What is the simplest collection of subsets of E which could not possibly be the set of all bases of $\mathcal{V}$ contained in E?

The first answer to our question might be $\emptyset$, for of course a vector space must have some bases. This answer is not very informative, so we ask for a nonempty collection of subsets of E. The next answer might be something like $\{\{a\},\{a,b\}\}$, for no basis can contain another. This answer too is

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not very informative, so we ask for a nonempty *clutter* of subsets of E, that is, is a family $\mathfrak{C}$ of subsets of E such that no element of $\mathfrak{C}$ contains any other. A natural answer to our question is any clutter which contains sets which have different numbers of elements; the simplest such clutter is $\mathfrak{C}_0 = \{\{a\}, \{b, c\}\}$. We make $\mathfrak{C}_0$ the central object of our exposition by defining a *matroid basis clutter* to be a nonempty clutter $\mathfrak{C}$ which does not yield any smaller clutters isomorphic to $\mathfrak{C}_0$.

To explain the term *yield* in the definition, we consider two ways that a clutter $\mathfrak{C} = \{\text{bases of a vector space } \mathcal{V} \text{ which are contained in } E\}$ might yield a smaller clutter which also consists of the bases of a vector space. If $S \subseteq E$ and $E \setminus S$ spans $\mathcal{V}$, then the bases contained in $E \setminus S$ will be the elements of $\mathfrak{C}$ which don't intersect S. Motivated by this, if $\mathfrak{C}$ is any clutter on a set E and $S \subseteq E$ then we define the *deletion* $\mathfrak{C} \setminus S$ to be the clutter $\{C \in \mathfrak{C} \mid C \subseteq E \setminus S\}$ on $E \setminus S$. Observe that the definition allows $\mathfrak{C} \setminus S = \emptyset$, just as in a vector space it is possible that $E \setminus S$ not span $\mathcal{V}$. Another way to derive a clutter on $E \setminus S$ from a clutter of bases of $\mathcal{V}$ is to consider the bases of the quotient space of $\mathcal{V}$ by the subspace spanned by S; they are the (images of) minimal sets T such that $S \cup T$ spans $\mathcal{V}$. (We cannot simply say "... sets T such that $S \cup T$ is a basis," for S may be linearly dependent.) This motivates a second definition: the *contraction* $\mathfrak{C}/S$ is $\{\text{minimal subsets } T \subseteq E \setminus S \mid S \cup T \text{ contains an element of } \mathfrak{C}\}$. A clutter which may be obtained from $\mathfrak{C}$ by a sequence of deletions and contractions is called a *minor* of $\mathfrak{C}$; one of them is $\mathfrak{C} = \mathfrak{C} \setminus \emptyset = \mathfrak{C}/\emptyset$. We leave it to the reader to verify that the minor operations satisfy the following three properties: for any $e_1 \neq e_2 \in E$, $(\mathfrak{C} \setminus e_1) \setminus e_2 = (\mathfrak{C} \setminus e_2) \setminus e_1$, $(\mathfrak{C}/e_1)/e_2 = (\mathfrak{C}/e_2)/e_1$ and $(\mathfrak{C}/e_1) \setminus e_2 = (\mathfrak{C} \setminus e_2)/e_1$. (It is customary to denote deletions and contractions of single elements $\mathfrak{C} \setminus e, \mathfrak{C}/e$ rather than $\mathfrak{C} \setminus \{e\}, \mathfrak{C}/\{e\}$.) It follows that the minors of $\mathfrak{C}$ are all of the form $(\mathfrak{C}/S_1) \setminus S_2 = (\mathfrak{C} \setminus S_2)/S_1$ for some disjoint $S_1, S_2 \subseteq E$.

A precise statement of our definition is then: *a matroid basis clutter is a nonempty clutter none of whose minors is isomorphic to* $\mathfrak{C}_0$. This definition has many immediate consequences, theorems about matroid basis clutters which reflect simple aspects of the structure of $\mathfrak{C}_0$. Here are three such theorems: if no minor of $\mathfrak{C}$ has an element of cardinality 1 then $\mathfrak{C}$ is a matroid basis clutter; if no minor of $\mathfrak{C}$ has an element of cardinality 2 then $\mathfrak{C}$ is a matroid basis clutter; and if no minor of $\mathfrak{C}$ has elements of different cardinalities then $\mathfrak{C}$ is a matroid basis clutter. These three statements are of fundamentally different types. "No minor of $\mathfrak{C}$ has an element of cardinality 1" is only possible if $\mathfrak{C} = \emptyset$ or $\{\emptyset\}$. "No minor of $\mathfrak{C}$ has an element of cardinality 2" is slightly more interesting, as it allows $\mathfrak{C}$ to have nonempty elements, though only singletons. "No minor of $\mathfrak{C}$ has elements of different cardinalities," though, turns out to be equivalent to our definition of a matroid basis clutter, as we shall see presently.

2. Some equivalent definitions

Indeed our general observation is that if P is almost any nontrivial property of bases of vector spaces which is not satisfied by $\mathfrak{C}_0$, then "no minor of $\mathfrak{C}$ violates P" turns out to be equivalent to the definition of a matroid basis clutter. Moreover, the different definitions of matroids that appear in the literature are almost all derivable from statements of the form "no minor of $\mathfrak{C}$ violates P" for some P familiar from linear algebra, and in some cases the P is surprisingly simple. In the rest of this note we discuss a variety of such re-definitions of matroids, beginning with the one already mentioned:

Proposition 2.1. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if no minor of $\mathfrak{C}$ has elements of different cardinalities.*

Proof. A clutter whose elements are all of the same cardinality is called *pure*. If every minor of $\mathfrak{C}$ is pure then $\mathfrak{C}_0$ is not a minor of $\mathfrak{C}$, and hence $\mathfrak{C}$ is a matroid basis clutter.

To prove the converse it is sufficient to prove that every non-pure clutter $\mathfrak{C}$ has a minor isomorphic to $\mathfrak{C}_0$. Suppose $\mathfrak{C}$ is a clutter on E and there are elements $C_1, C_2 \in \mathfrak{C}$ with $|C_1| < |C_2|$. Clearly $|E| \geq 3$ and if $|E| = 3$ then $\mathfrak{C}$ must be isomorphic to $\mathfrak{C}_0$. The proof proceeds by induction on $|E| \geq 4$. Replacing $\mathfrak{C}$ by $(\mathfrak{C} \setminus (E \setminus (C_1 \cup C_2)))/(C_1 \cap C_2)$ if necessary, we may presume that $E = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$.

Consider any element $e_1 \in C_1$. If $\mathfrak{C} \setminus e_1$ is not pure then the inductive hypothesis applies to $\mathfrak{C} \setminus e_1$, so it (and hence $\mathfrak{C}$) has a minor isomorphic to $\mathfrak{C}_0$. Otherwise $|C| = |C_2|\ \forall C \in \mathfrak{C} \setminus e_1$. If there is any $C \neq C_2 \in \mathfrak{C} \setminus e_1$ then $C \not\subseteq C_2 = E \setminus C_1$, so $C_1 \cap C \neq \emptyset$. Choose any $e \in C_1 \cap C$ and observe that $\mathfrak{C}/e$ is not pure, because $C_1 \setminus \{e\}, C \setminus \{e\} \in \mathfrak{C}/e$; apply the inductive hypothesis to $\mathfrak{C}/e$ to find a minor isomorphic to $\mathfrak{C}_0$. Similarly if $e_2 \in C_2$ and $\mathfrak{C} \setminus e_2$ is not pure, or $\mathfrak{C} \setminus e_2$ is pure and has any element other than C_1, then the inductive hypothesis provides a minor isomorphic to $\mathfrak{C}_0$.

It remains, then, to consider the possibility that $\mathfrak{C} \setminus e_1 = \{C_2\}$ for every $e_1 \in C_1$ and $\mathfrak{C} \setminus e_2 = \{C_1\}$ for every $e_2 \in C_2$, i.e., that $\mathfrak{C} = \{C_1, C_2\}$. Contracting all but one of the elements of C_1 and all but two of the elements of C_2 provides a minor of $\mathfrak{C}$ isomorphic to $\mathfrak{C}_0$. ■

Clearly a deletion of a pure clutter is pure, so Proposition 2.1 implies the simpler characterization: $\mathfrak{C}$ is a matroid basis clutter if and only if all its contractions are pure.

Exercise 1. Give an example of a pure, nonempty $\mathfrak{C}$ which is not a matroid basis clutter, that is, a pure $\mathfrak{C}$ which has a non-pure contraction.

Some answers to the exercises are given at the end of the paper.

Here is a simple property of vector spaces: if E is a spanning set of a vector space $\mathcal{V}$ then it is impossible to partition E into disjoint subsets so that every basis contained in E is contained in one of the subsets, unless $\mathcal{V}$ is of dimension ≤ 1 or E contains 0. (If $E = E_1 \cup E_2$ and $\mathcal{V}$ has bases

$B_1 \subseteq E_1$ and $B_2 \subseteq E_2$ then choose any $b_1 \in B_1$ and any $b_2 \in B_2$ not parallel to b_1; $\{b_1, b_2\}$ is independent and hence must be contained in a basis of $\mathcal{V}$.)

Definition. We say a clutter $\mathfrak{C}$ on E is *not significantly partitionable* if either every $C \in \mathfrak{C}$ has $|C| \leq 1$ or else there is no partition $E = E_1 \cup E_2$ such that every $C \in \mathfrak{C}$ is contained in E_1 or E_2 and each of E_1, E_2 contains at least one element of $\mathfrak{C}$.

Clearly $\mathfrak{C}_0$ does not satisfy this definition — it is significantly partitionable — so every nonempty clutter whose minors are all not significantly partitionable is a matroid basis clutter. The converse also holds:

Proposition 2.2. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if every minor of $\mathfrak{C}$ is not significantly partitionable.*

Proof. Suppose $\mathfrak{C}$ is a clutter which is significantly partitionable; then there are disjoint $E_1, E_2 \subseteq E$ such that every element of $\mathfrak{C}$ is contained in either E_1 or E_2, each of E_1, E_2 contains an element of $\mathfrak{C}$, and there is a $C_2 \in \mathfrak{C}$ with $C_2 \subseteq E_2$ and $|C_2| \geq 2$. Choose any $C_1 \in \mathfrak{C}$ with $C_1 \subseteq E_1$. The deletion $\mathfrak{C} \setminus (E \setminus (C_1 \cup C_2))$ is the clutter whose only elements are C_1 and C_2. Contracting all but one of the elements of C_1 and all but two of the elements of C_2 will yield a clutter isomorphic to $\mathfrak{C}_0$, so $\mathfrak{C}$ cannot be a minor of a matroid basis clutter. ■

Exercise 2. Give an example of a nonempty clutter $\mathfrak{C}$ which is not a matroid basis clutter but is not significantly partitionable.

Here is another simple property of vector spaces: the presence of one vector in a basis cannot generally be forced by the presence of another. That is, if b_1 and b_2 are distinct elements of a spanning set E of $\mathcal{V}$ and every basis contained in E which contains b_1 also contains b_2, then it must be that every basis contained in E contains b_2.

Definition. If $\mathfrak{C}$ is a clutter on E and $e_1 \neq e_2 \in E$ then we say e_1 *forces* e_2 if every element of $\mathfrak{C}$ containing e_1 also contains e_2, but not every element of $\mathfrak{C}$ contains e_2.

Proposition 2.3. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if no minor of $\mathfrak{C}$ has forced elements.*

Proof. $\mathfrak{C}_0 = \{\{a\}, \{b, c\}\}$ has forced elements, for b appears in no element of $\mathfrak{C}_0 \setminus c$. Hence a nonempty clutter whose minors have no forced elements must be a matroid basis clutter.

Suppose conversely that $\mathfrak{C}$ has a forced element; we claim that $\mathfrak{C}$ cannot be a minor of a matroid basis clutter. Suppose every element of $\mathfrak{C}$ which contains e_1 also contains e_2, and let $C_1, C_2 \in \mathfrak{C}$ have $e_1 \in C_1$ and $e_2 \notin C_2$. We claim that $C_2 \in \mathfrak{C}/e_1$. Certainly $C_2 \cup \{e_1\}$ contains an element of $\mathfrak{C}$, namely C_2. If S is a proper subset of C_2 then $S \cup \{e_1\}$ cannot contain any $C \in \mathfrak{C}$: if $e_1 \in C$ then $e_2 \in C$ but $e_2 \notin S \cup \{e_1\}$, and if $e_1 \notin C$ then $C \not\subseteq C_2$. This verifies the claim. Now observe that $C_1, C_2 \in \mathfrak{C}$ and $C_1 \setminus \{e_1\}, C_2 \in \mathfrak{C}/e_1$, so $\mathfrak{C}$ and $\mathfrak{C}/e_1$ cannot both be pure. ■

Exercise 3. Give an example of a nonempty clutter $\mathfrak{C}$ which is not a matroid basis clutter but has no forced elements.

If B_1 and B_2 are bases of a vector space $\mathcal{V}$ and $b_2 \in B_2 \setminus B_1$ then b_2 may be expressed as a linear combination of the elements of B_1, and if $b_1 \in B_1$ makes a nonzero contribution to this linear combination then a new basis of $\mathcal{V}$ may be obtained by removing b_2 from B_2 and replacing it with b_1.

Definition. A clutter $\mathfrak{C}$ has the *exchange property* if whenever $C_1, C_2 \in \mathfrak{C}$ and $c_2 \in C_2 \setminus C_1$, there is a $c_1 \in C_1 \setminus C_2$ with $(C_2 \setminus \{c_2\}) \cup \{c_1\} \in \mathfrak{C}$.

It is obvious that if $\mathfrak{C}$ satisfies the exchange property then it is not significantly partitionable and has no forced elements; consequently if all the minors of $\mathfrak{C}$ satisfy the exchange property then $\mathfrak{C}$ is a matroid basis clutter. For the fourth time, the converse also holds:

Proposition 2.4. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if all its minors have the exchange property.*

Proof. Suppose $\mathfrak{C}$ is a clutter which does not satisfy the exchange property; we claim that it isn't a matroid basis clutter. There are $C_1 \neq C_2 \in \mathfrak{C}$ and a $c_2 \in C_2 \setminus C_1$ such that $(C_2 \setminus \{c_2\}) \cup \{c_1\} \notin \mathfrak{C}\ \forall c_1 \in C_1$. Deleting $E \setminus (C_1 \cup C_2)$ if necessary, we may presume that $C_1 \cup C_2 = E$. Note that if $|E| \leq 2$ then our proposition is vacuously true, for there are no clutters which don't satisfy the exchange property; we proceed by induction on $|E| \geq 3$.

If there is any $C \neq C_1 \in \mathfrak{C}$ with $c_2 \notin C$, then choose an $e \in C_1 \setminus C$. $\mathfrak{C} \setminus e$ is a clutter on a smaller set than E, and it doesn't satisfy the exchange property, so the inductive hypothesis tells us $\mathfrak{C} \setminus e$ isn't a matroid basis clutter; hence $\mathfrak{C}$ isn't either.

Suppose there is no $C \neq C_1 \in \mathfrak{C}$ with $c_2 \notin C$; then $\mathfrak{C} \setminus c_2 = \{C_1\}$. If there is an $e \in E \setminus (C_1 \cup \{c_2\})$ then $e \in C_2$ but e appears in no element of $\mathfrak{C} \setminus c_2$, so $\mathfrak{C}$ has a forced element; hence $\mathfrak{C}$ is not a matroid basis clutter.

It remains to consider the possibility that $C_1 = E \setminus \{c_2\}$. If $\mathfrak{C}$ isn't pure then it is not a matroid basis clutter. If $\mathfrak{C}$ is pure, there is a $c_1 \in E$ such that $C_2 = E \setminus \{c_1\}$; then $(C_2 \setminus \{c_2\}) \cup \{c_1\} = C_1$, a contradiction. ■

At this point the reader may expect Exercise 4, but it is impossible to give an example of a clutter which satisfies the exchange property but isn't a matroid basis clutter. The reason is that satisfying the exchange property is fundamentally stronger than satisfying purity, not being significantly partitionable, or not having a forced element: if $\mathfrak{C}$ satisfies the exchange property then all its minors inherit the property from $\mathfrak{C}$. This is obviously true for deletions. It is not so obvious, but contractions of $\mathfrak{C}$ also inherit the exchange property. To prove this, we observe first that the exchange property has a simplifying effect on contractions. Suppose $\mathfrak{C}$ is a clutter on E which has the exchange property, and $e \in E$. If $C \in \mathfrak{C}/e$ then $C \cup \{e\}$ contains an element of $\mathfrak{C}$; this element must be either C or $C \cup \{e\}$, for otherwise a proper subset of C would be an element of $\mathfrak{C}/e$. Suppose

$C \in \mathfrak{C}/e$ and $C \in \mathfrak{C}$. If there is any element $C' \in \mathfrak{C}$ with $e \in C'$, then pick such a C' whose intersection with C is as large as possible. The exchange property implies that if $c' \in C' \setminus (C \cup \{e\})$ then there is a $c \in C \setminus C'$ such that $(C' \setminus \{c'\}) \cup \{c\} \in \mathfrak{C}$; this is an element of $\mathfrak{C}$ which contains e and has a larger intersection with C than C' has, contradicting the choice of C'. We conclude that there is no $c' \in C' \setminus (C \cup \{e\})$; that is, $C' \setminus C = \{e\}$. This contradicts the assumption that $C \in \mathfrak{C}/e$, for $C' \setminus \{e\}$ is a proper subset of C and $(C' \setminus \{e\}) \cup \{e\} = C' \in \mathfrak{C}$. ($C' \setminus \{e\}$ cannot equal C because $\mathfrak{C}$ is a clutter.) It follows that if $\mathfrak{C}$ has the exchange property and e appears in any element of $\mathfrak{C}$ then $\mathfrak{C}/e \subseteq \{C \mid C \cup \{e\} \in \mathfrak{C}\}$; the opposite inclusion is true for all clutters, so we conclude that if $\mathfrak{C}$ has the exchange property and e appears in any element of $\mathfrak{C}$ then $\mathfrak{C}/e = \{C \mid C \cup \{e\} \in \mathfrak{C}\}$.

Proposition 2.5. *If a nonempty clutter $\mathfrak{C}$ has the exchange property then so do all its contractions.*

Proof. Suppose $\mathfrak{C}$ has the exchange property and $e \in E$. If e appears in no element of $\mathfrak{C}$ then $\mathfrak{C}/e = \mathfrak{C}$ also has the exchange property. If e appears in some element of $\mathfrak{C}$ then as we just observed, $\mathfrak{C}/e = \{C \mid C \cup \{e\} \in \mathfrak{C}\}$. If $C_1, C_2 \in \mathfrak{C}/e$ and $c_2 \in C_2 \setminus C_1$ then $C_1 \cup \{e\}, C_2 \cup \{e\} \in \mathfrak{C}$ and $c_2 \in C_2 \cup \{e\} \setminus C_1 \cup \{e\}$; the exchange property of $\mathfrak{C}$ implies that there is a $c_1 \in C_1 \setminus C_2$ with $(C_2 \cup \{e\} \setminus \{c_2\}) \cup \{c_1\} \in \mathfrak{C}$. Then $(C_2 \setminus \{c_2\}) \cup \{c_1\} \in \mathfrak{C}/e$.

This shows that all the singleton contractions $\mathfrak{C}/e$ inherit the exchange property from $\mathfrak{C}$. To verify the proposition for a non-singleton contraction $\mathfrak{C}/S$, simply contract S one element at a time. ■

It follows that if $\mathfrak{C}$ satisfies the exchange property then all its minors also satisfy the exchange property. Consequently it is unnecessary to mention minors in Proposition 2.4; the phrase "all its minors have the exchange property" can be replaced by "it has the exchange property" without compromising the validity of the proposition. The exchange property is the most commonly used axiom for matroid basis clutters, and from our point of view the reason for its popularity is precisely that it can be used without mentioning minors; indeed, if one were to set out intentionally to strengthen purity, the absence of a significant partition, or the absence of forced elements to get a property that is inherited by minors, the exchange property would likely result. But there is a price to pay: the exchange property is more complicated than the others.

Recall that if $\mathfrak{C}$ has the exchange property and e appears in any element of $\mathfrak{C}$ then $\mathfrak{C}/e = \{C \mid C \cup \{e\} \in \mathfrak{C}\}$; we say that such clutters have *simple contractions*. Having simple contractions is another property of matroid basis clutters which is not shared by $\mathfrak{C}_0$, so it provides another characterization of matroid basis clutters:

Proposition 2.6. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if all its minors have simple contractions.*

Having simple contractions is not inherited by minors — consider that

$\mathfrak{C}_0$ is a contraction of $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c, d\}\}$ — so this characterization cannot be simplified so as to avoid mentioning minors.

3. Independent sets

In linear algebra, an independent set is simply a subset of a basis. When working with a clutter $\mathfrak{C}$, then, it seems natural to consider the subsets of the elements of $\mathfrak{C}$ to be the independent sets of $\mathfrak{C}$.

We would like to find a property of the independent sets of $\mathfrak{C}_0$ which could not apply to the linearly independent sets contained in a spanning set of a vector space. One that comes to mind is that $\mathfrak{C}_0$ has an independent set which is not of maximal size but which is not contained in any larger independent set, namely $\{a\}$. This is essentially a re-phrasing of the observation that $\mathfrak{C}_0$ is not pure, so we do not consider it in detail. The other one-element independent sets of $\mathfrak{C}_0$ are contained in a two-element "basis," but their relationship with the "basis" $\{a\}$ is peculiar: neither $\{b\}$ nor $\{c\}$ may be enlarged using an element of $\{a\}$ without losing its independence. In a vector space, of course, an independent set which is not a basis may be made into a larger independent set by adjoining *any* element not in its span. So we are led to the following

Definition. The independent sets of a clutter $\mathfrak{C}$ satisfy the *augmentation property* if whenever I_1, I_2 are independent sets of $\mathfrak{C}$ and $|I_1| < |I_2|$ there is an $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\}$ is independent.

The augmentation property is one of the standard axioms of matroid theory. It obviously implies the exchange property: just take $I_1 = C_2 \setminus \{c_2\}$ and $I_2 = C_1$. The converse implication is not quite so obvious, but it is not difficult. If $\mathfrak{C}$ satisfies the exchange property and I_1, I_2 are independent sets of $\mathfrak{C}$ with $|I_1| < |I_2|$ then there are $C_1, C_2 \in \mathfrak{C}$ with $I_1 \subseteq C_1$ and $I_2 \subseteq C_2$. If we choose them so that $|C_1 \cap C_2|$ is as large as possible then $C_1 \setminus I_1 \subseteq C_2$, for any element of $C_1 \setminus (I_1 \cup C_2)$ may be exchanged for an element of C_2; then $|C_1 \setminus I_1| > |C_2 \setminus I_2|$ implies that $C_1 \setminus I_1$ contains an element of I_2.

By the way, Proposition 2.1 frequently appears in the literature in a modified form: a nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if for every $S \subseteq E$ the maximal independent sets of $\mathfrak{C}$ contained in S are all of the same cardinality. What may be surprising about this form of Proposition 2.1 is that it refers to a deletion operation — the independent sets which intersect the complement of S are ignored — whereas in Proposition 2.1 it is the contractions, not the deletions, that are crucial. The key to the relationship between the two statements is in Proposition 2.6; we leave the details to the reader.

4. Circuits

We define a *circuit* of a clutter $\mathfrak{C}$ to be a minimal set $S \subseteq E$ such that no element of $\mathfrak{C}$ contains S. The word "circuit" may not seem relevant to linear algebra (it is motivated by the fact that a circuit in a graph is not contained in any tree), but certainly the idea of a minimal dependent set of vectors is natural: it is a set of vectors each of which can be expressed in a unique way as a linear combination of the others.

The circuits of $\mathfrak{C}_0 = \{\{a\}, \{b, c\}\}$ are $\{a, b\}$ and $\{a, c\}$. If these had arisen from a vector space $\mathcal{V}$, we would interpret them as asserting that a and b are parallel nonzero vectors and a and c are also parallel nonzero vectors; we would expect that consequently b and c would be parallel. That is, vector spaces have *transitive parallelism* — if $\{a, b\}$ and $\{a, c\}$ are circuits then $\{b, c\}$ is also a circuit — and $\mathfrak{C}_0$ does not. This yields another characterization of matroids.

Proposition 4.1. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if every minor of $\mathfrak{C}$ has transitive parallelism.*

Proof. If every minor of $\mathfrak{C}$ has transitive parallelism then $\mathfrak{C}_0$ isn't a minor of $\mathfrak{C}$. To prove the converse we must show that if $\mathfrak{C}$ does not have transitive parallelism then it is not a matroid basis clutter. Suppose $\{a, b\}$ and $\{a, c\}$ are circuits of $\mathfrak{C}$ but $\{b, c\}$ is not; then $\{a\}$ and $\{b, c\}$ are independent sets of $\mathfrak{C}$ but $\{a, b\}$ and $\{a, c\}$ are not, so $\mathfrak{C}$ cannot satisfy the augmentation property. ■

Exercise 4. Give an example of a nonempty clutter $\mathfrak{C}$ which has transitive parallelism but is not a matroid basis clutter.

A related property of matroid basis clutters is that they have *transitive circuits*: if a, b appear together in a circuit and a, c appear together in a circuit then b, c also appear together in a circuit. We will not duplicate here the proof that matroids have transitive circuits; it can be found in any textbook in the field. $\mathfrak{C}_0$ certainly violates this property, so we conclude that a nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if every minor of $\mathfrak{C}$ has transitive circuits.

Exercise 5. Give an example of a nonempty clutter $\mathfrak{C}$ which has transitive circuits but is not a matroid basis clutter.

Another striking aspect of the circuits of $\mathfrak{C}_0$ is that they have a common element. Suppose E is a spanning set of a vector space $\mathcal{V}$ and $e \in E$ is an element of every minimal linearly dependent subset of E. Then $E \setminus \{e\}$ is linearly independent, and hence e is expressible in a unique way as a linear combination of the elements of $E \setminus \{e\}$; if $S = \{$elements of E which make nonzero contributions to this linear combination$\}$ then $S \cup \{e\}$ is the only minimal dependent subset E can have. That is, if E is a spanning set of a vector space and the minimal linearly dependent subsets of E have a common element, then E only has one minimal linearly dependent subset.

Definition. A clutter $\mathfrak{C}$ has *non-intersecting circuits* if either it has no more than one circuit or it has more than one circuit and their intersection is empty.

Proposition 4.2. *A nonempty clutter $\mathfrak{C}$ is a matroid basis clutter if and only if every minor of $\mathfrak{C}$ has non-intersecting circuits.*

Proof. If every minor of $\mathfrak{C}$ has non-intersecting circuits then $\mathfrak{C}_0$ is not a minor of $\mathfrak{C}$ and hence $\mathfrak{C}$ is a matroid basis clutter.

To prove the converse we must show that if $\mathfrak{C}$ has intersecting circuits then it is not a matroid basis clutter. Suppose e is an element of all the circuits of $\mathfrak{C}$, and that $\mathfrak{C}$ has at least two circuits. $E \setminus \{e\}$ cannot contain any circuit of $\mathfrak{C}$, so it must be that $E \setminus \{e\}$ is contained in an element of $\mathfrak{C}$. If $E \in \mathfrak{C}$ then $\mathfrak{C}$ has no circuits; hence $E \setminus \{e\} \in \mathfrak{C}$. If $\mathfrak{C}$ is pure then every element of $\mathfrak{C}$ is $E \setminus \{x\}$ for some $x \in E$, and $\{x \mid E \setminus \{x\} \in \mathfrak{C}\}$ is clearly the only circuit of $\mathfrak{C}$, a contradiction. ■

Exercise 6. Give an example of a nonempty clutter $\mathfrak{C}$ which has non-intersecting circuits but is not a matroid basis clutter.

Considering the relationship between the exchange property and the simpler properties discussed in Section 2, it is natural to ask whether or not it is possible to find a strengthened version of transitive parallelism or non-intersecting circuits which will be inherited by minors. To frame such a strengthened version, let us consider for a moment the relationship between circuits of $\mathfrak{C}$ and circuits of a deletion $\mathfrak{C} \setminus S$. If $\gamma \subseteq E \setminus S$ is a circuit of $\mathfrak{C}$ then γ is not contained in any element of $\mathfrak{C}$, and hence is not contained in any element of $\mathfrak{C} \setminus S$; it follows that γ must contain a circuit of $\mathfrak{C} \setminus S$. (In general γ need not be a circuit of $\mathfrak{C} \setminus S$.) If $\mathfrak{C}$ has non-intersecting circuits then whenever γ_1 and γ_2 are two circuits of $\mathfrak{C}$ which have an element x in common, $\mathfrak{C}$ must have a circuit γ_3 which does not contain x, for otherwise x will appear in the intersection of the circuits of $\mathfrak{C}$. It might happen that γ_1 and γ_2 contain circuits of $\mathfrak{C} \setminus S$ which also have x in common. How can we arrange that γ_3 (or a circuit contained in γ_3) will be inherited by $\mathfrak{C} \setminus S$? The most direct way is to require $\gamma_3 \subseteq \gamma_1 \cup \gamma_2$, so that whenever γ_1 and γ_2 give rise to circuits of $\mathfrak{C} \setminus S$, so will γ_3. The resulting property (if $\gamma_1 \neq \gamma_2$ are circuits of $\mathfrak{C}$ and $x \in \gamma_1 \cap \gamma_2$ then $\mathfrak{C}$ has a circuit $\gamma_3 \subseteq (\gamma_1 \cup \gamma_2) \setminus \{x\}$) is called *circuit elimination*, and it is one of the standard axioms of matroid theory.

Proposition 4.3. *A nonempty clutter $\mathfrak{C}$ satisfies circuit elimination if and only if the independent sets of $\mathfrak{C}$ have the augmentation property.*

Proof. Suppose $\mathfrak{C}$ satisfies circuit elimination, I_1 and I_2 are independent sets of $\mathfrak{C}$ with $|I_1| < |I_2|$, and $I_1 \cup \{i_2\}$ is dependent for every $i_2 \in I_2 \setminus I_1$; we may presume that I_1 and I_2 have been chosen so that $|I_1 \setminus I_2|$ is as small as possible. For every $i_2 \in I_2 \setminus I_1$ there must be a circuit $\gamma_{i_2} \subseteq I_1 \cup \{i_2\}$. Necessarily $i_2 \in \gamma_{i_2}$, for the independent set I_1 cannot contain a circuit. If $|I_1 \setminus I_2| = 0$ then $I_1 \subset I_2$, so $I_1 \cup \{i_2\}$ is independent for every $i_2 \in I_2$, a contradiction. If $|I_1 \setminus I_2| = 1$ then the one element of $I_1 \setminus I_2$ must appear in every γ_{i_2}, because a singleton $\{i_2\}$ cannot be a circuit. As $|I_1| < |I_2|$, $|I_2 \setminus I_1| > |I_1 \setminus I_2|$ and hence there are $i_2 \neq i'_2 \in I_2 \setminus I_1$. Then γ_{i_2} and $\gamma_{i'_2}$ must both contain the one element of $I_1 \setminus I_2$; circuit elimination provides a circuit $\gamma \subseteq (\gamma_{i_2} \cup \gamma_{i'_2}) \setminus (I_1 \setminus I_2) \subseteq I_2$, an impossibility.

Proceeding inductively, suppose $|I_1 \setminus I_2| > 1$ and $i \in I_1 \setminus I_2$. As $|(I_1 \setminus \{i\}) \setminus I_2| < |I_1 \setminus I_2|$, the inductive hypothesis provides an $i_2 \in I_2 \setminus I_1$ such that $(I_1 \setminus \{i\}) \cup \{i_2\}$ is independent; as $|((I_1 \setminus \{i\}) \cup \{i_2\}) \setminus I_2| < |I_1 \setminus I_2|$ the inductive hypothesis also provides an $i'_2 \in I_2 \setminus (I_1 \cup \{i_2\})$ such that $(I_1 \setminus \{i\}) \cup \{i_2, i'_2\}$ is independent. It follows that $(I_1 \setminus \{i\}) \cup \{i_2, i'_2\}$ cannot contain any circuit, and hence $i \in \gamma_{i_2} \cap \gamma_{i'_2}$. Circuit elimination then provides a circuit $\gamma \subseteq (\gamma_{i_2} \cup \gamma_{i'_2}) \setminus \{i\} \subseteq (I_1 \setminus \{i\}) \cup \{i_2, i'_2\}$, contradicting the independence of $(I_1 \setminus \{i\}) \cup \{i_2, i'_2\}$.

The converse is very simple. Suppose $\mathfrak{C}$ satisfies the augmentation property, γ_1 and γ_2 are distinct circuits of $\mathfrak{C}$, $x \in \gamma_1 \cap \gamma_2$, and $\gamma_1 \cup \gamma_2 \setminus \{x\}$ is independent. Then $\gamma_1 \cap \gamma_2$ is a proper subset of γ_1 and hence an independent set of $\mathfrak{C}$, and repeatedly applying the augmentation property shows that $\gamma_1 \cup \gamma_2$ has an independent subset which contains $\gamma_1 \cap \gamma_2$ and is as large as $\gamma_1 \cup \gamma_2 \setminus \{x\}$. Such a subset must contain γ_1 or γ_2 and hence cannot be independent, a contradiction. ■

There are other characterizations of matroid basis clutters using circuits [5, 6]. For instance, it turns out that $\mathfrak{C}$ is a matroid basis clutter if and only if its circuits are preserved by deletion, i.e., for every $S \subseteq E$ the circuits of $\mathfrak{C} \setminus S$ are the circuits of $\mathfrak{C}$ that don't intersect S.

5. Summary

"Matroids" were introduced by Whitney [9] as a means of generalizing properties of vector spaces, though equivalent definitions have been given by others for other reasons; see [2] for a discussion of the history of the field. The fact that matroids may be defined in many different ways is one of their distinguishing characteristics, and is mentioned in most introductory treatments [2, 3, 7, 8, 10]; it is such a striking characteristic that matroid theorists coined the adjective *cryptomorphic* for axiom systems that determine different aspects of the same structure. The result that the class of matroid basis clutters is determined by a single forbidden minor is part of the folklore of the field. Its first clear statement in print may have been in [1], but it was mentioned in passing in [4].

The fundamental idea of our exposition is that these two characteristics of matroid theory — the multiplicity of axiomatizations and the existence of a single forbidden minor — are connected with each other: every property of the single forbidden minor gives rise to a theorem of the form "If no minor of $\mathfrak{C}$ has that property then $\mathfrak{C}$ is a matroid basis clutter," and surprisingly many of these theorems have valid converses and hence actually characterize matroids. We have presented several of these *minor matroid axioms* and the interested reader will readily find more by contrasting the structure of $\mathfrak{C}_0$ with other aspects of the structure of vector spaces, e.g., the rank function that assigns to a subset $S \subseteq E$ the dimension of the subspace it spans, and the span operator that assigns to S the set of all elements of E that can be expressed as linear combinations of the elements of S. It might seem strange that we attempt to give a simple exposition by introducing new axioms to an already large collection, but we hope that by providing motivation for the large number of axioms, giving examples of minor axioms, and noting that the major axioms of the field are simply versions of minor axioms that have been strengthened enough that deletions and contractions inherit them, we provide a coherent picture of the foundations of matroid theory.

Figure 1

6. Some answers to the exercises

1. $\{\{a, b\}, \{c, d\}\}$
2. $\{\{a, b, c\}, \{c, d\}\}$ or $\{\{a, b\}, \{b, c\}, \{c, d\}\}$ or $\{\{a, b\}, \{c, d\}, \{b, c, e\}\}$
3. $\{\{a\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ or $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c, d\}\}$
4. $\{\{a, b\}, \{a, c, d\}, \{b, c, d\}\}$ or $\{\{a, b\}, \{c, d\}, \{a, c, e\}, \{b, d, e\}\}$
5. $\{\{a\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ or $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c, d\}\}$
6. $\{\{a, b\}, \{c, d\}\}$ or $\{\{a, b\}, \{c\}, \{d\}\}$

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