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A Subset Expansion of the Coloured Tutte Polynomial

LORENZO TRALDI

Department of Mathematics, Lafayette College, Easton, Pennsylvania 18042, USA
(e-mail: traldil@lafayette.edu)

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Bollobás and Riordan introduce a Tutte polynomial for coloured graphs and matroids in [3]. We observe that this polynomial has an expansion as a sum indexed by the subsets of the ground-set of a coloured matroid, generalizing the subset expansion of the Tutte polynomial. We also discuss similar expansions of other contraction–deletion invariants of graphs and matroids.

1. The coloured Tutte polynomial

The *Tutte polynomial* or *dichromate* is an invariant of graphs and matroids which has been found to be associated with a variety of important and seemingly unrelated matters – vertex colourings, acyclic orientations and flows in graphs, reliability of communication networks, statistical mechanics and knot theory. In several of these instances it is natural to modify the Tutte polynomial to reflect weights associated with the matroid elements or graph edges being considered; for instance, consider the reliability of a network whose edges function with various probabilities, or an invariant of a knot diagram in which it is necessary to distinguish overpassing arcs from underpassing arcs. In [3] Bollobás and Riordan construct a 'universal' Tutte polynomial of weighted and coloured graphs, and mention that the invariant is easily extended to matroids. The purpose of this note is to observe that part of the theory of the ordinary (unweighted) Tutte polynomial generalizes to their universal invariant. Although we can direct the reader to several sources of background information – [1, 7] for graph theory, [5, 9] for matroid theory, and [1, 7, 8, 10] for the Tutte polynomial – [3] is our primary reference.

For the sake of convenience, we discuss matroids rather than graphs in this section; in Section 2 below we briefly discuss other contraction–deletion invariants, including non-matroidal invariants of graphs. Suppose (M, c) is a *coloured matroid*, *i.e.*, a matroid M on a finite set E together with a function c mapping E into some set Λ of *colours*. Let $\mathbb{Z}_\Lambda$ be the polynomial ring $\mathbb{Z}[\{X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda\}]$. Then Bollobás and Riordan define $W(M,c)$ to be an element of a quotient ring $\mathbb{Z}_\Lambda/I_0$, calculated by a recursion which involves removing the elements of E one at a time as follows.

Definition 1. If $e \in E$ is a *loop* of M and $\lambda = c(e)$ then $W(M,c) = Y_\lambda W(M-e,c)$; if $e \in E$ is an *isthmus* of M and $\lambda = c(e)$ then $W(M,c) = X_\lambda W(M/e,c)$; and if $e \in E$ is neither a loop nor an isthmus of M and $\lambda = c(e)$ then $W(M,c) = y_\lambda W(M-e,c) + x_\lambda W(M/e,c)$. The initial condition of the recursion is that the empty matroid $\emptyset$ has $W(\emptyset,c) = 1$.

It does not suit the purpose of this brief note to give a detailed summary of [3], but we recall several facts. Bollobás and Riordan do not actually discuss $W(M,c)$ in detail; instead they focus on a closely related invariant $W(G,c) = \alpha_{|V(G)|} \cdot W(M,c)$ of coloured graphs, where M is the cycle matroid of G. The purpose of considering the quotient ring $\mathbb{Z}_\Lambda/I_0$ is to ensure that $W(M,c)$ is independent of the order in which the elements of E are removed. Bollobás and Riordan prove that $W(M,c)$ is universal among matroid invariants satisfying recursions similar to that of Definition 1; this result is related to earlier work of Zaslavsky [12].

Suppose E is ordered as $E = \{e_1, \ldots, e_m\}$ and we perform the recursive calculation of Definition 1 by removing e_m first, then e_{m-1}, and so on until finally e_1 is removed. The result is a formula for $W(M,c)$ as a sum; we associate with each term of the sum the subset of E consisting of those elements which are contracted in obtaining that term. Then we have a term for each subset $B \subseteq E$ such that when one contracts the elements of B and deletes the elements of $E-B$, no elements of B become loops and no elements of $E-B$ become isthmuses. It turns out that these subsets B of E are precisely the bases of M. If B is a basis of M then an element $e_i \in B$ which contributes a power of X_λ to this term must be an isthmus at the time of its contraction; equivalently, i must be the least index of any element of the unique bond contained in $(E-B) \cup \{e_i\}$. Such an element of B is *internally active with respect to* B. Similarly, if $e_i \notin B$ contributes a power of Y_λ to the term, then e_i must be a loop at the time of its deletion, so i must be the least index of any element of the unique circuit contained in $B \cup \{e_i\}$; such an $e_i \notin B$ is *externally active with respect to* B. We see that the recursive application of Definition 1 results in the following.

Definition 2. For each basis B of M let $IA(B)$ and $EA(B)$ be the subsets of E consisting of the elements which are internally and externally active with respect to B, respectively. Then

$$W(M,c) = \sum_B \left(\prod_{e \in IA(B)} X_\lambda \right) \left(\prod_{e \in B - IA(B)} x_\lambda \right) \left(\prod_{e \in EA(B)} Y_\lambda \right) \left(\prod_{e \in E - B - EA(B)} y_\lambda \right),$$

with $\lambda = c(e)$ throughout.

Bollobás and Riordan prove in [3] that, in addition to being universal with respect to Definition 1, $W(M,c)$ is universal among invariants that possess *coloured activities expansions* like that of Definition 2.

By the way, it is not necessary to implement Definition 1 according to a fixed order of E. For instance, after e_m is removed there is nothing in Definition 1 which requires that the same element e_{m-1} be removed from both M/e_m and $M - e_m$. In general an implementation of Definition 1 follows some binary tree, which records the deletions and contractions that are performed. Activities may be defined with respect to the binary tree instead of a fixed order, and the resulting version of Definition 2 is verified just as in the discussion above. This modified version of Definition 2 is not crucial in the present discussion, but we shall see in Section 2 that activities defined with respect to binary trees are valuable when considering contraction–deletion invariants other than $W(M,c)$.

Definitions 1 and 2 are coloured versions of well-known descriptions of the Tutte polynomial, discussed in the introduction of [3]. The Tutte polynomial may also be described as a sum indexed by the subsets of E; this description is the *subset expansion* or *co-rank–nullity expansion* and is closely related to the formula for the *Whitney–Tutte dichromatic polynomial* mentioned in the introduction of [3]. Bollobás and Riordan do not give an analogous definition for $W(M,c)$, though they do observe in Sections 4 and 5 of [3] that some evaluations of $W(M,c)$ have subset expansions.

Following [4], we adopt the convention that $M - e = M/e$ whenever e is a loop or isthmus of M. It is this convention which makes it convenient for us to use matroids rather than graphs: the deletion and contraction of an isthmus from a graph do not result in the same graph, though there is certainly a very simple relationship between the two.

Definition 1′. $W(M,c)$ may be calculated recursively using these three reduction properties: if $e \in E$ is an isthmus of M and $\lambda = c(e)$ then $W(M,c) = (X_\lambda - a)W(M/e,c) + aW(M-e,c)$ for any a; if $e \in E$ is a loop of M and $\lambda = c(e)$ then $W(M,c) = aW(M-e,c) + (Y_\lambda - a)W(M/e,c)$ for any a; and if $e \in E$ is neither a loop nor an isthmus of M and $\lambda = c(e)$ then $W(M,c) = y_\lambda W(M-e,c) + x_\lambda W(M/e,c)$. The initial condition of the recursion is that $W(\emptyset,c) = 1$.

Different ways of choosing a will lead to different *generalized activities expansions* of $W(M,c)$; see [4] for examples of such expansions of the Tutte polynomial. Especially interesting is the choice of $a = X_\lambda - x_\lambda$ for an isthmus with $c(e) = \lambda$ and $a = y_\lambda$ for a loop with $c(e) = \lambda$. An implementation of Definition 1′ which involves removing the elements of E according to some order or binary tree, and using these choices of a, leads to the following subset expansion for $W(M,c)$.

Theorem 1.1.

$$W(M,c) = \sum_{S \subseteq E} \left(\prod_{e \in EI(S)} (X_\lambda - x_\lambda) \right) \left(\prod_{e \in E-S-EI(S)} y_\lambda \right) \left(\prod_{e \in IL(S)} (Y_\lambda - y_\lambda) \right) \left(\prod_{e \in S-IL(S)} x_\lambda \right).$$

The term of the sum indexed by a subset $S \subseteq E$ is the term obtained by deleting the elements of $E - S$ and contracting the elements of S. $EI(S)$ consists of the elements of $E - S$ which are deleted as isthmuses in obtaining this term, and $IL(S)$ consists of the elements of S which are contracted as loops.

Like Definitions 1 and 2, Theorem 1.1 does not make it obvious that $W(M,c)$ is independent of the order in which the elements of E are removed during the recursion. Also, the connection between $W(M,c)$ and the matroid structure of M is hidden in the definitions of $EI(S)$ and $IL(S)$. These features are more easily seen in two corollaries of Theorem 1.1.

Corollary 1.2. *Let $\lambda_0 \in \Lambda$ and let Z_{λ_0} be X_{λ_0} or Y_{λ_0}. Then*

$$x_{\lambda_0}^{|E|} y_{\lambda_0}^{|E|} Z_{\lambda_0} W(M,c) = Z_{\lambda_0} \sum_{S\subseteq E} \left(\prod_{e\in S} x_\lambda\right)\left(\prod_{e\in E-S} y_\lambda\right) x_{\lambda_0}^{|E|-|S|+r(S)} y_{\lambda_0}^{|E|-r(E)+r(S)} \times \left(X_{\lambda_0} - x_{\lambda_0}\right)^{r(E)-r(S)} \left(Y_{\lambda_0} - y_{\lambda_0}\right)^{|S|-r(S)}.$$

Proof. The definition of the ideal I_0 [3] implies that $Z_{\lambda_0}(X_\lambda - x_\lambda) y_{\lambda_0} = Z_{\lambda_0}(X_{\lambda_0} - x_{\lambda_0}) y_\lambda$ and $Z_{\lambda_0}(Y_\lambda - y_\lambda) x_{\lambda_0} = Z_{\lambda_0}(Y_{\lambda_0} - y_{\lambda_0}) x_\lambda$ for each $\lambda \in \Lambda$. Applying these identities repeatedly, we deduce from Theorem 1.1 that

$$x_{\lambda_0}^{|E|} y_{\lambda_0}^{|E|} Z_{\lambda_0} W(M,c) = Z_{\lambda_0} \sum_{S\subseteq E} \left(\prod_{e\in S} x_\lambda\right)\left(\prod_{e\in E-S} y_\lambda\right) x_{\lambda_0}^{|E|-|IL(S)|} y_{\lambda_0}^{|E|-|EI(S)|} \times \left(X_{\lambda_0} - x_{\lambda_0}\right)^{|EI(S)|} \left(Y_{\lambda_0} - y_{\lambda_0}\right)^{|IL(S)|}.$$

Corollary 1.2 now follows from Theorem 1 of [4], which states that $|EI(S)| = r(E) - r(S)$ and $|IL(S)| = |S| - r(S)$ for every $S \subseteq E$. □

Corollary 1.3 concerns an evaluation of $W(M,c)$, which Zaslavsky [12] calls the *normal function*. Let X and Y be individual indeterminates, let $\mathbb{Z}'_\Lambda$ be the polynomial ring $\mathbb{Z}[\{X,Y\} \cup \{x_\lambda, y_\lambda : \lambda \in \Lambda\}]$, and let $f : \mathbb{Z}_\Lambda \to \mathbb{Z}'_\Lambda$ be the homomorphism with $f(X_\lambda) = x_\lambda + (X-1)y_\lambda$ and $f(Y_\lambda) = y_\lambda + (Y-1)x_\lambda$ for each $\lambda \in \Lambda$.

Corollary 1.3.

$$f(W(M,c)) = \sum_{S\subseteq E} \left(\prod_{e\in S} x_\lambda\right)\left(\prod_{e\in E-S} y_\lambda\right) (X-1)^{r(E)-r(S)} (Y-1)^{|S|-r(S)}.$$

This evaluation of $W(M,c)$ is independent of the choice of an order on E, i.e., the ideal I_0 is contained in $\ker f$. *Moreover if $c : E \to \Lambda$ is injective then c and $f(W(M,c))$ determine M and $W(M,c)$.*

Proof. The assertion that $I_0 \subseteq \ker f$ may be verified directly from the definition of I_0 [3]. The formula for $f(W(M,c))$ follows from Corollary 1.2, after cancellation of $f(x_{\lambda_0}^{|E|} y_{\lambda_0}^{|E|} Z_{\lambda_0})$. Finally, the assertion that if $c : E \to \Lambda$ is injective then c and $f(W(M,c))$ determine M and $W(M,c)$ follows from the observation that if c is injective then one can use the coefficient $(\prod_{e\in S} x_\lambda)(\prod_{e\in E-S} y_\lambda)$ to single out the term of the sum that corresponds to a given $S \subseteq E$, and then one can use the exponent of either $X-1$ or $Y-1$ to determine $r(S)$. The rank function determines the matroid structure of M, and along with c, the matroid structure of M determines $W(M,c)$. □

2. Generalizations

Suppose M is a matroid on a finite set E, S is a subset of E, and we remove the elements of E one at a time in any order, contracting the elements of S and deleting the elements of $E-S$. Let $EI(S)$ be the set of elements of $E-S$ deleted as isthmuses during this process, and $IL(S)$ the set of elements of S contracted as loops. Theorem 1 of [4] tells us that then $|EI(S)|=r(E)-r(S)$ and $|IL(S)|=|S|-r(S)$. This implies a broad generalization of Theorem 1.1 above.

Any contraction–deletion calculation on a matroid or graph in which isthmuses and/or loops play a special role is tied to the matroid structure.

It may be surprising that it is not necessary that the result of the calculation be a matroid isomorphism invariant: a contraction–deletion recursion involving the edges of a graph is essentially tied to the cycle matroid even if the initial information of the recursion is non-matroidal. As we remarked in Section 1, it is also not necessary that the calculation rigidly follow an order of E; a calculation may follow any binary tree on E.

Consider for instance the most thoroughly studied contraction–deletion invariant of graphs, the chromatic polynomial $P(G)$. It can be determined through the following contraction–deletion calculation.

(1) If e is not a loop then $P(G)=P(G-e)-P(G/e)$.
(2) If e is a loop then $P(G)=0$.
(3) If $E(G)=\emptyset$ then $P(G)=\lambda^{|V(G)|}$.

If the definition is implemented according to some order or binary tree on $E(G)$ then the result is a formula for $P(G)$ as a sum indexed by the subsets $S\subseteq E(G)$ with the property that in the portion of the calculation corresponding to the contraction of elements of S and deletion of elements of $E-S$, no loop is ever encountered. That is, both $IL(S)$ and $EL(S)=\{$elements of $E-S$ which are deleted as loops in this part of the calculation$\}$ are empty. The resulting formula for $P(G)$,

$$P(G)=\sum_{\substack{IL(S)=\emptyset\\ EL(S)=\emptyset}}(-1)^{|S|}\lambda^{|V(G/S)|},$$

is the *broken circuits expansion* [11]. Observe that the sets S which contribute to the sum are determined by the cycle matroid, even though the contribution of each such S reflects non-matroidal information.

A more general invariant is the *universal V-function* $Z(G)$ [2, 6], which may be defined as follows; let $x_0, x_1, \ldots$ be independent indeterminates.

(1) If e is not a loop then $Z(G)=Z(G-e)+Z(G/e)$.
(2) If G has no non-loop edges and for each $v\in V(G)$ we let $L(v)=\{$loops of G incident on $v\}$ and $d(v)=|L(v)|$ then

$$Z(G)=\sum_{T\subseteq E(G)}\left(\prod_{v\in V(G)}x_{|T\cap L(v)|}\right),$$

or equivalently

$$Z(G) = \prod_{v \in V(G)} \left(\sum_{i=0}^{d(v)} \binom{d(v)}{i} x_i \right).$$

The definition cannot be implemented strictly following an order of $E(G)$, because part (1) of the definition cannot remove loops. Instead an implementation follows a binary tree whose leaves represent loop graphs (*i.e.*, graphs all of whose edges are loops). Part (1) of the definition yields an expansion indexed by the independent sets of the cycle matroid:

$$Z(G) = \sum_{IL(S)=\emptyset} Z(G/S).$$

Once again, the sum is indexed by sets which are determined by the cycle matroid, but the contribution of each set is not determined by the cycle matroid.

By the way, this expansion of $Z(G)$ connects two others which are already known. Using the second form of part (2) of the definition, this expansion yields the formula for $Z(G)$ that appears in Theorem 9 of [2]. If instead the first form of part (2) is used, then we associate the summand corresponding to $T \subseteq E(G/S)$ with $S \cup T$, and obtain an expansion of $Z(G)$ indexed by the subsets of $E(G)$; this expansion is the original definition of $Z(G)$ [6].

Corollary 1.3 may also be generalized.

Any invariant which results from a coloured contraction–deletion calculation on a matroid or graph will provide enough information to determine the matroid structure, if the calculation singles out isthmuses and/or loops for special attention and the colour map c is injective.

For instance, the coloured version of $Z(G)$ [2] is defined using the recursion: if e is not a loop then $Z_c(G) = y_{c(e)} Z_c(G-e) + x_{c(e)} Z_c(G/e)$. For our purposes it is necessary only to observe that this recursion defines $Z_c(G)$ as a linear function of the values of Z_c on loop graphs; we refer the reader to [2] for details about these values. If $c : E(G) \to \Lambda$ is injective and the various $x_{c(e)}$ and $y_{c(e)}$ are independent indeterminates then a recursive calculation of $Z_c(G)$ follows a binary tree just like one representing a calculation of $Z(G)$. Inspecting the tree we may identify, for each $S \subseteq E(G)$, the edges which become loops when non-loop elements of S are contracted and non-loop elements of $E(G) - S$ are deleted; consequently the binary tree determines $IL(S)$ and $EL(S)$ for each $S \subseteq E(G)$. This information is preserved in $Z_c(G)$, because c is injective. Theorem 1 of [4] tells us that $|IL(S)| = |S| - r(S)$ and consequently $Z_c(G)$ determines the rank of each $S \subseteq E(G)$. That is, even though $Z_c(G)$ is not a matroid invariant it does contain sufficient information to determine the cycle matroid; it also contains additional information which is non-matroidal, reflecting the values of Z_c on loop graphs.

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