ELSEVIER

Discrete Mathematics 248 (2002) 279–282

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Chain polynomials and Tutte polynomials

Lorenzo Traldi

Department of Mathematics, Lafayette College, Easton, PA 18042, USA

Received 4 December 2000; received in revised form 13 April 2001; accepted 2 July 2001

Abstract

The recently introduced *chain* and *sheaf polynomials* of a graph are shown to be essentially equivalent to a weighted version of the Tutte polynomial. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Chain polynomial; Tutte polynomial; Weighted matroid

In [3] Read and Whitehead introduced a multilinear polynomial invariant of a graph, the *chain polynomial*, which captures the effect on the chromatic polynomial of adjoining or removing vertices of degree two. In [5] the present author noted that a weighted version of the Tutte polynomial of a matroid may be used to capture the effect on the Tutte polynomial of removing elements which are parallel or in series. In this note, we observe that the polynomials studied in the two articles are closely related. We refer the reader to [6,7] for expositions of the properties of the Tutte polynomial (or *dichromate*) and other polynomial invariants of graphs and matroids.

The chain polynomial of [3] is associated with a multigraph G whose edges have been labeled with elements of a commutative ring with unity. (Two different conventions are actually followed in [3]; at first an edge is labeled with an integer n_a, but later, a power ω^{n_a} of the variable ω is replaced by a symbol a, and these symbols are treated as elements of a commutative ring with unity.) The chain polynomial may be defined using the chromatic polynomials of graphs obtained from G by inserting chains of degree-two vertices into its edges, or using the flow polynomials of subgraphs of G. For our purposes, however, it is most convenient to use the following recursive description of the chain polynomial in terms of deletions and contractions, extracted from Section 4 of [3]: if G has no edges then $\mathrm{Ch}(G) = 1$; if an edge of G is not a

E-mail address: traldil@lafayette.edu (L. Traldi).

0012-365X/02/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0012-365X(01)00350-8

loop and is labeled a then $\mathrm{Ch}(G)=(a-1)\mathrm{Ch}(G-a)+\mathrm{Ch}(G/a)$; and if a is the label of a loop in G then $\mathrm{Ch}(G)=(a-\omega)\mathrm{Ch}(G-a)$.

Let M be a doubly weighted matroid, i.e., a matroid M on a set E equipped with a pair of functions p,q mapping E into some commutative ring with unity. The Tutte polynomial of M is given by the formula

$$t(M)=\sum_{S\subseteq E}\left(\prod_{e\in S}p(e)\right)\left(\prod_{e\notin S}q(e)\right)(x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Weighted versions of the Tutte polynomial have been of use in statistical physics and associated fields like network reliability and knot theory (see [1,2,4]); p and q are often associated with the probabilities that certain elements of a system (hydrogen bonds between water molecules or communication links between nodes of a network, for instance), may or may not operate. For our purposes, it is convenient to use a recursive description of $t(M)$: $t(\emptyset)=1$; if e is a loop of M then $t(M)=(q(e)+(y-1)p(e))t(M-e)$; if e is an isthmus of M then $t(M)=(p(e)+(x-1)q(e))t(M/e)$; and if e is neither a loop nor an isthmus of M then $t(M)=p(e)t(M/e)+q(e)t(M-e)$. We refer the reader to [8] for a thorough discussion of weighted Tutte polynomials, including a characterization of invariants which satisfy identities like $t(M)=p(e)t(M/e)+q(e)t(M-e)$ when e is neither a loop nor an isthmus.

A couple of observations about weighted Tutte polynomials are worth making here. One observation is that it is not generally necessary to use two non-trivial weight functions: if we replace p and q by $p'\equiv p/q$ and $q'\equiv 1$ then the Tutte polynomial of the resulting doubly weighted matroid M' determines that of M, because

$$t(M)=\left(\prod_{e\in E}q(e)\right)t(M').$$

(Elements of E with $q(e)=0$ should be contracted before M' is constructed.)

Another observation is that the weighted Tutte polynomial can carry much more information than the ordinary Tutte polynomial does. For instance, suppose R is a polynomial ring with indeterminates in E, one of the functions p,q is identically 1, and the other of the functions p,q maps each $e\in E$ to itself. Then $t(M)$ is essentially a table of values for the rank function of M, i.e., the information contained in $t(M)$ is essentially equivalent to the information contained in the matroid structure of M. This can remain true even if we suppress one of the variables x,y in $t(M)$, either by evaluating that variable to a constant in R or by restricting our attention to the coefficient of a certain power of that variable in $t(M)$.

Comparing the recursive descriptions of $\mathrm{Ch}(G)$ and $t(M)$ we immediately deduce the following proposition, which also follows from Zaslavsky's characterization of strong Tutte functions of graphs and matroids [8].

Proposition 1. *Suppose G is an edge-labeled multigraph and M is the doubly weighted matroid on $E=E(G)$ obtained from the circuit matroid of G by letting the weights of an edge labeled a be $p(a)=1$ and $q(a)=a-1$. Then the chain polynomial* $\mathrm{Ch}(G)$ *is obtained from the Tutte polynomial $t(M)$ by evaluating $y=2-\omega$ and $x=2$.*

Note that if the edges of G are labeled with independent indeterminates then like the evaluations of $t(M)$ discussed immediately before Proposition 1, $\mathrm{Ch}(G)$ is essentially equivalent to the circuit matroid M. Indeed Proposition 1 gives the formula

$$\mathrm{Ch}(G) = \sum_{S \subseteq E} \left(\prod_{e \notin S} q(e) \right) (1-\omega)^{|S|-r(S)}$$

which makes it clear that $\mathrm{Ch}(G)$ and the rank function of M determine each other. (This formula for $\mathrm{Ch}(G)$ is also implicit in [3]; see for instance formula (2.5) there.)

Read and Whitehead briefly discuss a *sheaf polynomial* $\mathrm{Sh}(G)$ which is dual to the chain polynomial [3]; the sheaf polynomial captures the effect on the flow polynomial of adjoining or removing parallel edges, and can be described using the chromatic polynomials of subgraphs of G. We presume that the sentence "… we are led to define a 'sheaf' polynomial by omitting the factor before the summation in (7.10)" of p. 355 of [3] was intended to refer to their formula (7.9). Their formulas (7.9) and (7.5) then yield

$$(-1)^{n-1}\mathrm{Sh}(G) = (1-\omega)^{n-k} \sum_{S \subseteq E} (1-\omega)^{-n+k_S} \left(\prod_{e \in S} (a-1) \right)$$

in which a is the label of the edge e, k_S is the number of connected components of the subgraph $(V(G), S)$ of G, $k = k_E$ is the number of connected components of G itself, and $n = |V(G)|$. Combining powers of $1-\omega$ and recalling that the rank function of the circuit matroid of G is given by $r(S) = n - k_S$, we deduce that

$$(-1)^{n-1}\mathrm{Sh}(G) = \sum_{S \subseteq E} (1-\omega)^{r(E)-r(S)} \left(\prod_{e \in S} (a-1) \right),$$

which implies the following.

Proposition 2. *Suppose G is an edge-labeled multigraph and M is the doubly weighted matroid on $E = E(G)$ obtained from the circuit matroid of G by letting the weights of an edge labeled a be $p(a) = a - 1$ and $q(a) = 1$. Then the sheaf polynomial $\mathrm{Sh}(G)$ is obtained from the Tutte polynomial $t(M)$ by multiplying by $(-1)^{n-1}$ and evaluating $y = 2$ and $x = 2 - \omega$.*

Note the duality between Propositions 1 and 2: x and y are reversed, and so are p and q.

By the way, the chain and sheaf polynomials are also closely connected to the edge-weighted dichromatic polynomial $Q(G; t, z)$ we studied in [4]. We leave it to the interested reader to verify that if we change variables using $\omega = 1 - tz$ then transforming edge-labels into edge-weights using $w(e) = t/(a-1)$ (respectively, $w(e) = (a-1)/z$) will allow us to relate $\mathrm{Ch}(G)$ (respectively, $\mathrm{Sh}(G)$) to $Q(G; t, z)$.

References

[1] C.J. Colbourn, The Combinatorics of Network Reliability, Oxford University Press, Oxford, 1987.
[2] C.M. Fortuin, P.W. Kasteleyn, On the random-cluster model, Physica 57 (1972) 536–564.
[3] R.C. Read, E.G. Whitehead Jr., Chromatic polynomials of homeomorphism classes of graphs, Discrete Math. 204 (1999) 337–356.
[4] L. Traldi, A dichromatic polynomial for weighted graphs and link polynomials, Proc. Amer. Math. Soc. 106 (1989) 279–286.
[5] L. Traldi, Series and parallel reductions for the Tutte polynomial, Discrete Math. 220 (2000) 291–297.
[6] W.T. Tutte, Graph Theory, Cambridge University Press, Cambridge, 1984.
[7] N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, 1992.
[8] T. Zaslavsky, Strong Tutte functions of matroids and graphs, Trans. Amer. Math. Soc. 334 (1992) 317–347.