A conjecture about sums of disjoint products

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Abstract

A sum of disjoint products (SDP) representation of a Boolean function is useful because it makes readily available certain information about the function; however a typical SDP contains many more terms than an equivalent ordinary sum of products. We conjecture the existence of certain particular SDP forms of $x_1 + ... + x_t$, which could be used as patterns in creating relatively economical SDP forms of other Boolean functions.

Let $S = S(x_1, ..., x_n)$ be a Boolean function. A *sum of disjoint products* (or *SDP*) form of S is a formula

$$S(x_1, ..., x_n) = \sum_{j=1}^{m} \prod_{i=1}^{n} a_j(x_i)$$

with the following properties.

1. No product $\prod_{i=1}^{n} a_j(x_i)$ is 0.

2. Each $a_j(x_i)$ depends only on x_i; that is, $a_j(x_i) \in \{1, x_i, \bar{x}_i\}$.

3. If $j \neq j'$ then the products $\prod_{i=1}^{n} a_j(x_i)$ and $\prod_{i=1}^{n} a_{j'}(x_i)$ are logically disjoint, i.e., there is at least one $i \in \{1, ..., n\}$ such that $a_j(x_i)$ is the negation of $a_{j'}(x_i)$.

A typical SDP involves many more terms than an equivalent ordinary sum of products, but in compensation it makes a great deal of information readily available. For instance,

$$\sum_{j=1}^{m} 2^{|\{i \mid a_j(x_i)=1\}|}$$

is $\left|S^{-1}(\{1\})\right|$, the number of combinations of truth values of $x_1, ..., x_n$ which satisfy S. Because sums of disjoint products are so informative, they have been used in algorithms to calculate network reliability; see [1] for a survey.

If S is given as an ordinary sum of products $S = \sum_{s=1}^{t} S_s$, then the seemingly universal strategy for obtaining an SDP form of S involves using the SDP form

$$\sum_{s=1}^{t} x_s = x_1 + \sum_{s=2}^{t} (x_s \cdot \bar{x}_{s-1} \cdot ... \cdot \bar{x}_1) \tag{1}$$

as a pattern. The pattern is applied to S by finding SDP forms of the products $S_s \cdot \bar{S}_{s-1} \cdot ... \cdot \bar{S}_1$ one at a time. It will come as no surprise that the products $S_s \cdot \bar{S}_{s-1} \cdot ... \cdot \bar{S}_1$ with large s are generally more complicated than those with small s, and their SDP forms generally require more terms. Consequently if one is interested in finding relatively small SDP forms of Boolean functions then it seems worthwhile to investigate the various SDP forms of $\sum_{s=1}^{t} x_s$ which might be used instead of (1), with special attention to those which involve conjunctions of relatively few negations.

Theorem. *An SDP form*

$$\sum_{s=1}^{t} x_s = \sum_{j=1}^{m} \prod_{s=1}^{t} a_j(x_s)$$

of $\sum_{s=1}^{t} x_s$ *must include a product involving at least* $\left\lfloor \frac{t}{2} \right\rfloor$ *negations.*

Proof. For $j \in \{1, ..., m\}$ let $T_j = \{s \in \{1, ..., s\} \mid a_j(x_s) = x_s\}$ and $F_j = \{s \in \{1, ..., s\} \mid a_j(x_s) = \bar{x}_s\}$. Then the jth term of the SDP form is

$$\left(\prod_{s \in T_j} x_s\right) \cdot \left(\prod_{s \in F_j} \bar{x}_s\right),$$

and this term is satisfied by every assignment of truth values in which the elements of T_j are all true and the elements of F_j are all false.

For each particular $k \in \{1, ..., t\}$, $\sum_{s=1}^{t} x_s$ is satisfied when x_k is true and every x_s with $s \neq k$ is false. Consequently there is at least one $j_k \in \{1, ..., m\}$ with $k \notin F_{j_k}$ and $T_{j_k} \subseteq \{k\}$. T_{j_k} cannot be empty, because $\sum_{s=1}^{t} x_s$ is not satisfied when every x_s is false; hence $T_{j_k} = \{k\}$. Evidently $j_1, ..., j_m$ are pairwise distinct, as the T_{j_k} are pairwise distinct.

Suppose $|F_{j_k}| < \left\lfloor \frac{t}{2} \right\rfloor$ $\forall k \in \{1, ..., t\}$, and consider a particular $k \in \{1, ..., t\}$. According to condition 3 of the definition given at the beginning of the note, for each $k' \neq k \in \{1, ..., t\}$ there is at least one $s \in \{1, ..., t\}$ such that $a_{j_k}(x_s)$ is the negation of $a_{j_{k'}}(x_s)$; $T_{j_k} = \{k\}$ and $T_{j_{k'}} = \{k'\}$, so this can only happen if $k \in F_{j_{k'}}$ or $k' \in F_{j_k}$. Consequently k appears in at least $t - 1 - |F_{j_k}|$ of the sets $F_{j_{k'}}$ with $k' \neq k$; as $|F_{j_k}| \leq -1 + \left\lfloor \frac{t}{2} \right\rfloor$, it follows that k appears in at least $t - \left\lfloor \frac{t}{2} \right\rfloor$ of the sets $F_{j_{k'}}$ with $k' \neq k$. This is true for every $k \in \{1, ..., t\}$, so we conclude that

$$\sum_{k=1}^{t} |F_{j_k}| \geq t(t - \left\lfloor \frac{t}{2} \right\rfloor) \geq t \left\lfloor \frac{t}{2} \right\rfloor .$$

This contradicts the assumption that $|F_{j_k}| < \left\lfloor \frac{t}{2} \right\rfloor \; \forall k \in \{1, ..., t\}$. ■

Conjecture. *For every integer t there is an SDP form of $\sum_{s=1}^{t} x_s$ in which no product involves more than $\left\lfloor \frac{t}{2} \right\rfloor$ negations.*

We do not know of any systematic way to address the conjecture, so we have resorted to explicit constructions. The SDP

$$x_1 + x_2 + x_3 = x_1 x_2 x_3 + \bar{x}_1 x_2 + \bar{x}_2 x_3 + \bar{x}_3 x_1 \tag{2}$$

suggests a specialized form of the conjecture, namely that for prime p there is an SDP form of $\sum_{s=1}^{p} x_s$ which contains no product involving more than $\frac{p-1}{2}$ negations and is also symmetric with respect to cyclic permutation of $x_1, ..., x_p$. (The restriction to prime p arises from the assumption of symmetry, which implies that all the binomial coefficients $\binom{p}{k}$ with $0 < k < p$ are divisible by p.) Taking advantage of the assumed symmetry, which is a considerable convenience in the construction, we have been able to confirm the conjecture for every odd prime $p \leq 23$. These cyclically symmetric SDP forms can be quite large: (2) involves only four disjoint terms, but other examples we have found involve tens of terms ($p = 5$ or 7), hundreds of terms ($p = 11$ or 13), thousands of terms ($p = 17$ or 19), even hundreds of thousands of terms ($p = 23$). Nevertheless, the upper bound on the number of negations that appear in any one term implies that these patterns may be used to provide SDP forms of n-variable Boolean functions whose term counts are smaller than those of SDP forms which follow the conventional pattern (1) by factors up to $O(n^{11})$.

We refer the interested reader to [2] for more examples, including SDP forms with multiple-variable inversion. We close with our thanks to Alexandru O. Balan, who helped us find and understand some of these examples, and to Lafayette College, which supported this work.

References

[1] S. Rai, M. Veeraraghavan, and K. S. Trivedi, *A survey of efficient reliability computation using disjoint products approach*, Networks **25** (1995), 147-163.

[2] L. Traldi, *Non-minimal sums of disjoint products*, Reliability Engineering and System Safety **91** (2006), 533-538.