Test2

Wednesday, March 30, 2016 11:09 AM

Test2

Math 300

Test 2

April 1, 2016

Name: Key

You must show ALL of your work in order to receive credit.

If your scratch paper shows work that leads to your solution, please turn in in inside the test. Otherwise, dispose of it yourself.

1. Modified true/false: If the statement below is *always* true, write "true". Otherwise, correct the statement. (2 points each)

   (a) If V is a 5 dimensional vector space and U_1 and U_2 are subspaces, then $\dim(U_1) + \dim(U_2) \leq 5$.

   F: $\dim(U_1 + U_2) \leq 5$, or $\dim(U_1) \oplus \dim(U_2) \leq 5$

   (b) The ~~smallest~~ possible list of ~~independent~~ vectors in a 10 dimensional space contains 10 vectors.

   F: largest OR spanning

   (c) Let v be a vector in the n dimensional vector space V over $\mathbb{F}$. Then $(v)_B$ is a vector in V.

   F: $(v)_B \in \mathbb{F}^n$

   (d) If $T : V \to W$ is a linear transformation between finite dimensional vector spaces, then $T(\mathbf{0}_V) = \mathbf{0}_W$.

   True

   (e) Every list of vectors in a finite dimensional vector space V can be extended to a basis for V.

   F: vectors in list must be independent

2. Are the vectors

$$\begin{aligned} f(x) &= x^4 - x^3 + 5x^2 + 3x - 1 \\ g(x) &= 8x^4 - 3x^3 + 4x + 7 \\ h(x) &= -3x^4 + x^3 + x^2 - x - 3 \end{aligned}$$

in $\mathcal{P}_4(\mathbb{R})$ independent or dependent? Explain your answer carefully. (15 points)

Independent $\iff \alpha f + \beta g + \gamma h = 0$

$\implies \alpha = \beta = \gamma = 0$.

Row Reduce

$$\begin{pmatrix} 1 & 8 & -3 \\ -1 & -3 & 1 \\ 5 & 0 & 1 \\ 3 & 4 & -1 \\ -1 & 7 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 8 & -3 \\ 0 & 5 & -2 \\ 0 & -40 & 16 \\ 0 & -20 & 8 \\ 0 & 15 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 8 & -3 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

γ is a free variable, Infinitely many solutions.

Vectors are Not independent.

3. Recall that $C(-\infty,\infty)$ is the (infinite dimensional) vector space of all continuous real valued functions defined on $(-\infty,\infty)$. The vectors f_1, f_2, and $f_3 \in C(-\infty,\infty)$ given by $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, and $f_3(x) = 1$ span a finite dimensional subspace U.

(a) Find a basis for U. (10 points)

$\cos^2 x + \sin^2 x = 1$, but

$\alpha \cos^2 x + \beta \sin^2 x = 0 \Rightarrow \alpha = \beta = 0$,

so $(\cos^2 x, \sin^2 x)$

(b) Determine the dimension of U. (5 points)

$\dim(U) = 2$

4. Find a formula for the dimension of the vector space $\mathfrak{sl}(n, \mathbb{R})$ of all $n \times n$ trace 0 matrices. (15 points)

$$\mathfrak{sl}(2,\mathbb{R}): \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \to \dim. = 3$$

$$\mathfrak{sl}(3,\mathbb{R}): \begin{pmatrix} \alpha_1 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_2 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & -\alpha_1-\alpha_2 \end{pmatrix} \to \dim. = 8$$

in general, one entry is fixed, the other n^2-1 are free.

So $\dim(\mathfrak{sl}(n,\mathbb{R})) = n^2-1$.

5. Let f be a vector in the space $\mathcal{P}_3(\mathbb{R})$ of polynomials of degree at most 3. If f has coordinates

$$(f)_B = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

with respect to basis

$$B = (x^3 - 1, x^3 + 1, 2 - x^2, 4x^2 - x),$$

find $f(x)$. (15 points)

$$f(x) = -2(x^3 - 1) + (x^3 + 1) + 3(4x^2 - x)$$

$$= -x^3 + 12x^2 - 3x + 3.$$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined on the basis

$$B = \left(\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right)$$

$$\quad v_1 \qquad v_2 \qquad v_3$$

for $\mathbb{R}^3$ by

$$T\left(\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -6 \end{pmatrix}.$$

Find $T\left(\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right)$. (15 points)

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = -\frac{1}{2}v_1 - 2v_2 + \frac{1}{3}v_3$$

So $T\left(\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right) = -\frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

$$= \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

7. ~~Let $B = (v_1, v_2, \ldots, v_n)$ be a basis for a finite dimensional vector space V, and let u, w be any vectors in V, and $\alpha \in \mathbb{F}$. Recall that $(u)_B$ indicates the vector of coordinates of u with respect to B. Prove that $(u + w)_B = (u)_B + (w)_B$. (15 points)~~

Let $S = (v_1, \ldots, v_n)$ be a basis for U, and suppose that $\alpha_1, \ldots, \alpha_{n+1}$ are constants so that

$$\alpha_1 v_1 + \ldots + \alpha_n v_n + \alpha_{n+1} v = \vec{0}.$$

If $\alpha_{n+1} = 0$, then so do $\alpha_1, \ldots, \alpha_n$, since the v_i are independent. If $\alpha_{n+1} \neq 0$, then $v = \frac{-\alpha_1}{\alpha_{n+1}} v_1 - \ldots - \frac{\alpha_n}{\alpha_{n+1}} v_n$, so that $v \in U$, a contradiction.

So $\alpha_1 = \ldots = \alpha_n = \alpha_{n+1} = 0$, so that $S \cup (v)$ is independent.