Math 300
Test 2
April 1, 2016

Name: ____________________________

You must show ALL of your work in order to receive credit.

If your scratch paper shows work that leads to your solution, please turn in in inside the test. Otherwise, dispose of it yourself.
1. Modified true/false: If the statement below is always true, write “true”. Otherwise, correct the statement. (2 points each)

(a) If $V$ is a 5 dimensional vector space and $U_1$ and $U_2$ are subspaces, then $\dim(U_1) + \dim(U_2) \leq 5$.

F: \[ \dim(U_1 + U_2) \leq 5 \] or $\dim(U_1) + \dim(U_2) \leq 5$

(b) The smallest possible list of independent vectors in a 10 dimensional space contains 10 vectors.

F: \[ \text{largest spanning} \] OR \[ \text{OR} \]

(c) Let $v$ be a vector in the $n$ dimensional vector space $V$ over $F$. Then $(v)_B$ is a vector in $V$.

F: $(v)_B \in F^n$

(d) If $T: V \to W$ is a linear transformation between finite dimensional vector spaces, then $T(0_V) = 0_W$.

True

(e) Every list of vectors in a finite dimensional vector space $V$ can be extended to a basis for $V$.

F: Vectors in list must be independent
2. Are the vectors

\[
\begin{align*}
f(x) &= x^4 - x^3 + 5x^2 + 3x - 1 \\
g(x) &= 8x^4 - 3x^3 + 4x + 7 \\
h(x) &= -3x^4 + x^3 + x^2 - x - 3
\end{align*}
\]

in \( \mathbb{P}_4(\mathbb{R}) \) independent or dependent? Explain your answer carefully. (15 points)

Independent \( \iff \lambda f + \beta g + \gamma h = 0 \]

\[ \Rightarrow \lambda = \beta = \gamma = 0. \]

Row Reduce

\[
\begin{bmatrix}
1 & 8 & -3 \\
-1 & -3 & 1 \\
5 & 0 & 1 \\
3 & 4 & -1 \\
-1 & 7 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 8 & -3 \\
0 & 5 & -2 \\
0 & -10 & 16 \\
0 & -20 & 8 \\
0 & 15 & -6
\end{bmatrix}
\]

\( v \) is a free variable, \( \Rightarrow \)

Infinitely many solutions, vectors are not independent.
3. Recall that \( C(-\infty, \infty) \) is the (infinite dimensional) vector space of all continuous real valued functions defined on \((-\infty, \infty)\). The vectors \( f_1, f_2, \) and \( f_3 \in C(-\infty, \infty) \) given by \( f_1(x) = \cos^2 x, f_2(x) = \sin^2 x, \) and \( f_3(x) = 1 \) span a finite dimensional subspace \( U \).

(a) Find a basis for \( U \). (10 points)

\[
\cos^2 x \perp \sin^2 x = 1, \quad \text{but}
\]

\[
d \cos^2 x + \beta \sin^2 x = 0 \Rightarrow \lambda = \beta = 0,
\]

so \((\cos^2 x, \sin^2 x)\)

(b) Determine the dimension of \( U \). (5 points)

\[\dim(U) = 2\]
4. Find a formula for the dimension of the vector space \( \text{sl}(n, \mathbb{R}) \) of all \( n \times n \) trace 0 matrices. (15 points)

\[
\text{sl}(2, \mathbb{R}): \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \text{dim.} = 3
\]

\[
\text{sl}(3, \mathbb{R}): \begin{pmatrix} a & b_{12} & b_{13} \\ b_{21} & a & b_{23} \\ b_{31} & b_{32} & a \end{pmatrix} \rightarrow \text{dim.} = 8
\]

In general, one entry is fixed, the other \( n^2 - 1 \) are free.

So \( \text{dim} (\text{sl}(n, \mathbb{R})) = n^2 - 1 \).
5. Let $f$ be a vector in the space $\mathcal{P}_3(\mathbb{R})$ of polynomials of degree at most 3. If $f$ has coordinates

$$
(f)_B = \begin{pmatrix}
-2 \\
1 \\
0 \\
3
\end{pmatrix}
$$

with respect to basis $B = (x^3 - 1, x^3 + 1, 2 - x^2, 4x^2 - x)$, find $f(x)$. (15 points)

$$
f(x) = -2(x^3 - 1) + (x^3 + 1) + 3(4x^2 - x)
$$

$$
= -x^3 + 12x^2 - 3x + 3
$$
6. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be the linear transformation defined on the basis

\[
B = \left( \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \right) = \left( \begin{array}{c} 2 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 3 \end{array} \right)
\]

for \( \mathbb{R}^3 \) by

\[
T \left( \begin{array}{c} 2 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 4 \\ -2 \end{array} \right)
\]

\[
T \left( \begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right) = \left( \begin{array}{c} 2 \\ 0 \end{array} \right)
\]

\[
T \left( \begin{array}{c} 0 \\ 0 \\ 3 \end{array} \right) = \left( \begin{array}{c} 0 \\ -6 \end{array} \right)
\]

Find \( T \left( \begin{array}{c} -1 \\ 2 \end{array} \right) \). (15 points)

\[
\left( \begin{array}{c} -1 \\ 2 \end{array} \right) = -\frac{1}{2} \mathbf{v}_1 - 2 \mathbf{v}_2 + \frac{1}{3} \mathbf{v}_3
\]

So

\[
T \left( \begin{array}{c} -1 \\ 2 \\ -1 \end{array} \right) = -\frac{1}{2} \left( \begin{array}{c} 4 \\ 1 \\ 0 \end{array} \right) - 2 \left( \begin{array}{c} 2 \\ 0 \\ 3 \end{array} \right) + \frac{1}{3} \left( \begin{array}{c} 6 \\ -6 \end{array} \right)
\]

\[
= \left( \begin{array}{c} -6 \\ -1 \end{array} \right)
\]
7. Let \( B = \{v_1, v_2, \ldots, v_n\} \) be a basis for a finite dimensional vector space \( V \), and let \( u, w \) be any vectors in \( V \), and \( \alpha \in \mathbb{F} \). Recall that \( (u)_{B} \) indicates the vector of coordinates of \( u \) with respect to \( B \). Prove that \((u + w)_{B} = (u)_{B} + (w)_{B}\). (15 points)

Let \( S = (u_1, \ldots, u_n) \) be a basis for \( U \), and suppose that \( d_1, d_2, \ldots, d_n \) are constants so that
\[
d_1u_1 + \cdots + d_nu_n + d_{n+1}v = 0.
\]
If \( d_{n+1} = 0 \), then so do \( d_1, \ldots, d_n \), since the \( u_i \) are independent. If \( d_{n+1} \neq 0 \), then \( v = -\frac{d_1}{d_{n+1}}u_1 - \cdots - \frac{d_n}{d_{n+1}}u_n \), so that \( v \notin U \), a contradiction.

So \( d_1 = \cdots = d_n = d_{n+1} = 0 \), so that \( SU(v) \) is independent.