1. Find the coordinates of each of $f_2(x) = x^2$, $f_1(x) = x$, and $f_0(x) = 1$ with respect to each of the following bases for $\mathcal{P}_2(\mathbb{R})$:

   (a) $B_1 = (x^2,\ x,\ 1)$
   Solution: We have

$$\begin{aligned} f_2(x) &= x^2 + 0x + 0 \\ f_1(x) &= 0x^2 + x + 0 \\ f_0(x) &= 0x^2 + 0x + 1, \end{aligned}$$

   so the desired coordinates are

$$\begin{aligned} (f_2)_{B_1} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (f_1)_{B_1} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ (f_0)_{B_1} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

   (b) $B_2 = (2x - 3,\ x^2 + 1,\ 2x^2 - x)$
   Solution: We have

$$\begin{aligned} f_2(x) &= \frac{1}{7}(2x-3) + \frac{3}{7}(x^2+1) + \frac{2}{7}(2x^2 - x) \\ f_1(x) &= \frac{2}{7}(2x-3) + \frac{6}{7}(x^2+1) - \frac{3}{7}(2x^2 - x) \\ f_0(x) &= -\frac{1}{7}(2x-3) + \frac{4}{7}(x^2+1) - \frac{2}{7}(2x^2 - x), \end{aligned}$$

so the desired coordinates are

$$
\begin{aligned}
(f_2)_{B_2} &= \frac{1}{7}\begin{pmatrix}1\\3\\2\end{pmatrix}\\
(f_1)_{B_2} &= \frac{1}{7}\begin{pmatrix}2\\6\\-3\end{pmatrix}\\
(f_0)_{B_2} &= \frac{1}{7}\begin{pmatrix}-1\\4\\-2\end{pmatrix}.
\end{aligned}
$$

2. The vectors

$$e_{11} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \; e_{12} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \text{ and } e_{21} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

form a basis B for the space $\mathfrak{sl}(2,\mathbb{R})$ of 2×2 trace 0 matrices with real entries. We may extend this basis to a basis $\hat{B}$ for all of $\mathcal{M}_2(\mathbb{R})$ by adjoining vector

$$e_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

(a) Find the coordinates of

$$v = \begin{pmatrix}7 & -3\\ 1 & -7\end{pmatrix}$$

as a vector in $\mathfrak{sl}(2,\mathbb{R})$ with respect to

$$B = (e_{11},\ e_{12},\ e_{21}).$$

Solution: Since

$$v = 7e_{11} - 3e_{12} + e_{21},$$

the coordinates are given by

$$(v)_B = \begin{pmatrix}7\\-3\\1\end{pmatrix}.$$

(b) Find the coordinates of v *as a vector in* $\mathcal{M}_2(\mathbb{R})$ with respect to $\hat{B} = (e_{11},\ e_{12},\ e_{21},\ e_1)$.
Solution: We don't actually have to use e_1 to build v, so the coordinates for v with respect to basis $\hat{B}$ are

$$(v)_{\hat{B}} = \begin{pmatrix}7\\-3\\1\\0\end{pmatrix}.$$

3. Let V be a finite dimensional vector space over $\mathbb{F}$ with basis

$$B = (v_1,\ v_2,\ \ldots,\ v_n).$$

Prove the following assertions:

(a) $(u+v)_B = (u)_B + (v)_B$ for all $u,\ v \in V$
Solution: Given

$$\begin{aligned} u &= \alpha_1 v_1 + \ldots + \alpha_n v_n \\ v &= \beta_1 v_1 + \ldots + \beta_n v_n \\ u+v &= (\alpha_1+\beta_1)v_1 + \ldots + (\alpha_n+\beta_n)v_n \end{aligned}$$

so that

$$\begin{aligned} (u)_B &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \\ (v)_B &= \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \\ (u+v)_B &= \begin{pmatrix} \alpha_1+\beta_1 \\ \alpha_2+\beta_2 \\ \vdots \\ \alpha_n+\beta_n \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}, \end{aligned}$$

it is clear that

$$(u+v)_B = (u)_B + (v)_B.$$

(b) $(\lambda u)_B = \lambda(u)_B$ for all $u \in V$, $\lambda \in \mathbb{F}$.
Solution: Given

$$u = \alpha_1 v_1 + \ldots + \alpha_n v_n \text{ and } \lambda u = (\lambda\alpha_1)v_1 + \ldots + (\lambda\alpha_n)v_n$$

so that

$$(u)_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ and } (\lambda u)_B = \begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \\ \vdots \\ \lambda\alpha_n \end{pmatrix} = \lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

it is clear that

$$(\lambda u)_B = \lambda (u)_B.$$

4. For each of the following functions, determine if the function is a linear transformation on the given spaces (please justify your conclusions):

(a) $S : \mathbb{R}^2 \to \mathbb{R}$,

$$S\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \sqrt{x^2 + y^2}.$$

Solution: Not a linear transformation. For example, setting

$$u_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so that

$$u_1 + u_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

we have

$$\begin{aligned} T(u_1) &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ T(u_2) &= \sqrt{1 + 1} \\ &= \sqrt{2} \\ T(u_1 + u_2) &= \sqrt{9 + 9} \\ &= \sqrt{18} \\ &\neq \sqrt{2} + \sqrt{20}. \end{aligned}$$

(b) $D : \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$,

$$D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = ad - bc.$$

Solution: Not a linear transformation. For example, let

$$X_1 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so that

$$X_1 + X_2 = \begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}.$$

Now clearly

$$\begin{aligned} D(X_1) &= 0 \\ D(X_2) &= 0 \\ D(X_1 + X_2) &= 2. \end{aligned}$$

(c) For fixed $b \in \mathbb{R}^3$, $P_b : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by projection onto b, that is

$$P_b(a) = \left(\frac{a \cdot b}{|b|^2}\right) b.$$

Solution: P_b is a linear transformation. Given

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \ v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

so that

$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \text{ and } \lambda v_1 = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \end{pmatrix}$$

we have

$$\begin{aligned} P_b(v_1) &= \left(\frac{v_1 \cdot b}{|b|^2}\right) b \\ &= \left(\frac{x_1 b_1 + y_1 b_2 + z_1 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\ P_b(v_2) &= \left(\frac{v_2 \cdot b}{|b|^2}\right) b \\ &= \left(\frac{x_2 b_1 + y_2 b_2 + z_2 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\ P_b(v_1 + v_2) &= \left(\frac{(v_1 + v_2) \cdot b}{|b|^2}\right) b \\ &= \left(\frac{(x_1 + x_2) b_1 + (y_1 + y_2) b_2 + (z_1 + z_2) b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\ &= \left(\frac{(x_1 b_1 + y_1 b_2 + z_1 b_3) + (x_2 b_1 + y_2 b_2 + z_2 b_3)}{b_1^2 + b_2^2 + b_3^2}\right) b \\ &= \left(\frac{x_1 b_1 + y_1 b_2 + z_1 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b + \left(\frac{x_2 b_1 + y_2 b_2 + z_2 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\ &= P_b(v_1) + P_b(v_2). \end{aligned}$$

Similarly,

$$\begin{aligned}
P(\lambda v_1) &= \left(\frac{\lambda v_1 \cdot b}{|b|^2}\right) b \\
&= \left(\frac{\lambda x_1 b_1 + \lambda y_1 b_2 + \lambda z_1 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\
&= \left(\frac{\lambda (x_1 b_1 + y_1 b_2 + z_1 b_3)}{b_1^2 + b_2^2 + b_3^2}\right) b \\
&= \lambda \left(\frac{x_1 b_1 + y_1 b_2 + z_1 b_3}{b_1^2 + b_2^2 + b_3^2}\right) b \\
&= \lambda P_b(v_1).
\end{aligned}$$

Thus P_b is a linear transformation.