Invertible Transformations and Isomorphic Vector Spaces

Certain types of linear transformations are particularly important: in this section, we will be interested in transformations that are "reversible". Thus we record the following definition:

Definition 3.53. A linear transformation $T : V \to W$ is called *invertible* if there is another linear transformation $S : W \to V$ so that $ST : V \to V$ is the identity map on V and $TS : W \to W$ is the identity map on W. S is called an *inverse* of T.

Example. Let $T : \mathcal{U}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Then T is invertible, and $S : \mathbb{R}^3 \to \mathcal{U}_2(\mathbb{R})$ defined by

$$S\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}$$

is clearly an inverse, since

$$TS = T_I$$

and

$$ST = T_I$$

(where the first T_I is the identity transformation on $\mathbb{R}^3$, and the second is the identity transformation on $\mathcal{U}_2(\mathbb{R})$).

Example. It is easy to see that the linear transformation

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left(\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}$$

has no inverse. Indeed, there is no way to recover information about v_3 after π has been applied.

If a transformation T has an inverse S, it is quite easy to see that S is the *only* such map, as indicated by the following theorem:

Theorem 3.54. If T is invertible, then its inverse is unique, and is denoted by T^{-1}.

Proof. *Exercise.*

Looking forward, we would like to have some criteria that will allow us to easily determine if a transformation has an inverse; indeed, we wish to characterize the set of invertible maps. The following theorem provides us with that characterization:

Theorem 3.56. A linear transformation T is invertible if and only if T is injective and surjective.

Proof. If $T : V \to W$ is invertible, then $T^{-1}T$ is the identity map on V, and TT^{-1} is the identity map on W. We wish to show that T is injective and surjective.

To show that T is injective, let v be any vector in $\operatorname{null}(T)$, so that $T(v) = \mathbf{0}$. Now $T^{-1}T(v) = v$ since $T^{-1}T$ is the identity on V; however, it is easy to see that, for any linear transformation S, $S(\mathbf{0}) = \mathbf{0}$. In particular, $T^{-1}(\mathbf{0}) = \mathbf{0}$; so $v = \mathbf{0}$, and T is injective.

To show that T is also surjective, let w be any vector in W. Since TT^{-1} is the identity on W, we know that $TT^{-1}(w) = w$. In particular $v := T^{-1}(w)$ is a vector in V, and is in fact the unique vector in V so that $T(v) = w$. Therefore $\operatorname{range}(T) = W$, and T is surjective.

On the other hand, suppose that T is both injective and surjective. Since T is surjective, every vector $w \in W$ is the image of some $v \in V$. Since T is injective, there is *precisely one* such $v \in V$ for every $w \in W$. Thus the map $S : W \to V$, defined by setting $S(w) = v$ if $T(v) = w$, is well defined.

We wish to show that $S = T^{-1}$, so consider the maps ST and TS. Given any $v \in V$, $ST(v) = S(T(v)) = S(w)$, and by definition $S(w) = v$, so that $ST(v) = v$. ST is thus the identity map on V. Similarly, for $w \in W$, $TS(w) = T(S(w)) = T(v)$, where v is the unique element of V with $T(v) = w$. Thus $TS(w) = w$ and TS is the identity on W.

Finally, we must show that the map S is a linear transformation; so consider $S(w_1 + w_2)$. Now each of w_1 and w_2 is the image of a unique element of V, say v_1 and v_2 respectively. Indeed,

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2,$$

so $v_1 + v_2$ is the unique element of V whose image is $w_1 + w_2$. By definition,

$$\begin{aligned} S(w_1 + w_2) &= v_1 + v_2 \\ &= S(w_1) + S(w_2). \end{aligned}$$

Now consider $S(\lambda w)$; clearly if v is the unique element of V so that $T(v) = w$, then by the linearity of T, λv is the unique element of V so that $T(\lambda v) = \lambda w$. Thus

$$\begin{aligned} S(\lambda w) &= \lambda v \\ &= \lambda S(v). \end{aligned}$$

S is therefore linear, and we conclude that $S = T^{-1}$.

If there is an invertible linear transformation between a pair V and W of vector spaces, then there is a sense in which the spaces are the same (up to the particular decoration in the space). Indeed, each vector in V is matched with one and only one vector in W (and vice-versa), and the operations work exactly the same way in V as in W. With this in mind, we record the following definition:

Definition 3.58. If there is an invertible linear transformation T between a pair V and W of vector spaces, then V and W are called *isomorphic*, denoted by

$$V \simeq W.$$

T is an *isomorphism* between the two spaces.

Example. Since the linear transformation $T : \mathcal{U}_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

is invertible, T is actually an isomorphism, and $\mathcal{U}_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic spaces.

Example. The linear transformation

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left(\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}$$

is surjective, but *not* injective (and so not invertible). However, we *cannot* conclude that $\mathcal{M}_2(\mathbb{R})$ are $\mathcal{U}_2(\mathbb{R})$ are not isomorphic. Indeed, there *could be* some other linear transformation between the two spaces that is an isomorphism. However, the next theorem will rule out this possibility.

The following theorem characterizes isomorphic finite dimensional vector spaces:

Theorem 3.59. Finite dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Proof. Suppose that V and W are isomorphic via linear transformation $T : V \to W$. Then $\text{null}\,(T) = \{\mathbf{0}\}$ and $\text{range}\,(T) = W$. Let

$$B = (v_1,\ v_2,\ \ldots,\ v_n)$$

be a basis for V; I claim that

$$B' = (T(v_1),\ T(v_2),\ \ldots,\ T(v_n))$$

is a basis for W. Every $w \in W$ is the image of precisely one $v \in V$, $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$. Then

$$\begin{aligned} T(v) &= T(\alpha_1 v_1 + \ldots + \alpha_n v_n) \\ &= \alpha_1 T(v_1) + \ldots + \alpha_n T(v_n) \\ &= w. \end{aligned}$$

Thus every $w \in W$ is a linear combination of the vectors in B', so span $(B') = W$.

If α_i are chosen so that $\alpha_1 T(v_1) + \ldots + \alpha_n T(v_n) = \mathbf{0}$, then since $T^{-1}(\mathbf{0}) = \mathbf{0}$, we have

$$\begin{aligned} \mathbf{0} &= T^{-1}(\mathbf{0}) \\ &= T^{-1}(\alpha_1 T(v_1) + \ldots + \alpha_n T(v_n)) \\ &= \alpha_1 v_1 + \ldots + \alpha_n v_n. \end{aligned}$$

The vectors in B are independent since B is a basis, so $\alpha_i = 0$ for all i; thus the vectors in B' are independent as well, and B' is a basis for W. We conclude that $\dim(V) = \dim(W)$.

On the other hand, suppose that V and W are finite dimensional with the same dimension. Let $B = (v_1,\ v_2,\ \ldots,\ v_n)$ and $B' = (w_1,\ w_2,\ \ldots,\ w_n)$ be any bases for V and W respectively. Define the map T on B by $T(v_i) = w_i$ for all i, and extend T linearly to all of V. T is a linear transformation by construction (see the proof of Theorem 3.5), and we wish to show that T is an isomorphism. To do so, we need simply find an inverse for T. So let $S : W \to V$ be the map defined on B' by by $S(w_i) = v_i$ for all i, and extend S linearly to all of W. Again, S is linear, and we need to show that TS and ST are both the identity map.

For any $v = \alpha_1 v_1 + \ldots + \alpha_n v_n \in V$, we have

$$\begin{aligned} ST(v) &= S(T(v)) \\ &= S(T(\alpha_1 v_1 + \ldots + \alpha_n v_n)) \\ &= S(\alpha_1 T(v_1) + \ldots + \alpha_n T(v_n)) \\ &= S(\alpha_1 w_1 + \ldots + \alpha_n w_n) \\ &= \alpha_1 S(w_1) + \ldots + \alpha_n S(w_n) \\ &= \alpha_1 v_1 + \ldots + \alpha_n v_n \\ &= v, \end{aligned}$$

so that ST is the identity on V. The proof that TS is the identity on W is identical, mutatis mutandis, so that $S = T^{-1}$, and V and W are isomorphic.

Example. We can now conclude that $\mathcal{M}_2(\mathbb{R})$ are $\mathcal{U}_2(\mathbb{R})$ are not isomorphic: $\dim(\mathcal{M}_2(\mathbb{R})) = 4$, while $\dim(\mathcal{U}_2(\mathbb{R})) = 3$.

Two interesting corollaries follow immediately from the theorem:

Corollary. Every n dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

Corollary 3.60. Let V and W be n and m dimensional vector spaces over $\mathbb{F}$, respectively. Then the vector space $\mathcal{L}(V, W)$ of all linear transformations from V to W is isomorphic to the vector space $\mathcal{M}_{m,n}(\mathbb{F})$ of all $m \times n$ matrices.

Proof. *Exercise.*

We conclude the discussion of this section with a theorem:

Theorem 3.69. Let $T : V \to V$ be a linear operator on the (finite dimensional) vector space V. Then the following are equivalent:

(a) T is an isomorphism.

(b) T is injective.

(c) T is surjective.

Proof. Clearly (a) implies (b).

To show that (b) implies (c), recall that the Fundamental Theorem of Linear Transformations says that

$$\dim(V) = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T)).$$

Since $\dim(\operatorname{null}(T)) = 0$, we have

$$\dim(V) = \dim(\operatorname{range}(T)),$$

so that $V = \operatorname{range}(T)$, and T is surjective.

To show that (c) implies (b), we again use the Fundamental Theorem:

$$\dim(V) = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T)),$$

and since $\operatorname{range}(T) = V$,

$$\dim(V) = \dim(\operatorname{range}(T))$$

so that $\dim(\operatorname{null}(T)) = 0$. Thus T is injective (as well as surjective), so (c) also implies (a), and the conditions are equivalent.