Null Space and Range

As we work to understand linear transformations in more detail, we pause to consider two subspaces that can give us a wealth of information about the transformation itself: the null space and the range.

Null Space of a Transformation

The first subset of interest is the null space:

**Definition 3.12.** Let \( T : V \to W \) be a linear transformation. The null space of \( T \), denoted by \( \text{null}(T) \), is the set of all vectors \( v \in V \) so that \( T(v) = 0 \).

**Remark.** Notice that the nullspace is a subset of the domain \( V \), and *not* of the codomain \( W \). To find \( \text{null}(T) \), we must determine which vectors in \( V \) are mapped to \( 0 \in W \).

**Example.** Given *any* vector spaces \( V \) and \( W \), the transformation \( T_0 : V \to W \),

\[
T_0 : V \to W,
\]

given by

\[
T_0(u) = 0,
\]

has null space

\[
\text{null}(T_0) = V,
\]

since *every* vector in \( V \) maps to \( 0 \) under the action of \( T_0 \).

**Example.** Let \( V \) be *any* vector space, and let

\[
T_I : V \to V
\]

be defined by

\[
T_I(v) = v.
\]

Then since the *only* vector mapped to \( 0 \) is \( 0 \) itself, we see that

\[
\text{null}(T_I) = \{0\}.
\]
Example. Given linear transformation
\[ \pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R}) \]
defined by
\[ \pi \left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}, \]
find null (\( \pi \)).

The null space of \( \pi \) is the set of all vectors in \( \mathcal{M}_2(\mathbb{R}) \) that are mapped to
\[ 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{U}_2(\mathbb{R}). \]

If
\[ \pi \left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]
then we must have
\[ v_1 = v_2 = v_4 = 0. \]

However, \( v_3 \) is free, so the null space of \( \pi \) is the set
\[ \text{null} (\pi) = \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \mid v \in \mathbb{R} \right\}. \]

Example. With
\[ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \]
find the null space of the map \( T_M : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by
\[ T_M \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]

Now null (\( T_M \)) is the set of all vectors \( v \in \mathbb{R}^2 \) so that
\[ Mv = 0. \]

Of course, to find the set of all solutions to the matrix equation above, we may simply row reduce \( M \) itself:
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
so that there is only one solution $v$ to

$$Mv = 0.$$  

Thus

$$\text{null (} T_M \text{)} = \{ 0 \}.$$

As you may have already guessed, $\text{null (} T \text{)}$ is more than just a subset of $V$; it is actually a subspace of $V$, as indicated by the following theorem:

**Theorem 3.14.** If $T : V \rightarrow W$ is a linear transformation, then $\text{null (} T \text{)}$ is a subspace of $V$.

We will not prove the theorem in detail, as it is quite straightforward. Indeed, the standard procedure applies immediately here: one must show that, if $v, w \in \text{null (} T \text{)}$, then $v + w \in \text{null (} T \text{)}$ as well (i.e., show that $T(v + w) = 0$ if $T(v) = 0$ and $T(w) = 0$), and that $\lambda v \in \text{null (} T \text{)}$ (i.e., show that $T(\lambda v) = 0$ if $T(v) = 0$). Both tasks are easily accomplished using the fact that $T$ is a linear transformation.

We have seen examples of transformations with null space $\{ 0 \}$; null space $V$; or even null space “in between” $0$ and $V$. We will soon see that transformations with null space $\text{null (} T \text{)} = \{ 0 \}$ are particularly nice. Accordingly, we introduce the following definition:

**Definition 3.15.** A linear transformation $T : V \rightarrow W$ is one-to-one or injective if

$$T(u) = T(v) \rightarrow u = v.$$

**Remark.** Injectivity simply means that no two different vectors map to the same place. The idea looks a bit like the concept of invertibility from calculus; while there is a relation, we will see that invertibility and injectivity are not quite the same, and you should not assume that injectivity implies any sort of “invertibility”.

The idea of an injective map is actually closely tied to the null space of that transformation, as indicated by the following theorem:

**Theorem 3.16.** The linear transformation $T : V \rightarrow W$ is injective if and only if $\text{null (} T \text{)} = \{ 0 \}$.

**Proof.** For any linear transformation $T$,

$$T(0) = 0.$$  

Thus if $T$ is injective, then no other vector $v$ can map to $0$, so that

$$\text{null (} T \text{)} = \{ 0 \}.$$
Unit 3, Section 3: Null Space and Range

On the other hand, suppose that $T$ has null space $\{0\}$, and let $u$ and $v$ be any vectors in $V$ so that $T(u) = T(v)$. Then $T(u) - T(v) = 0$. However, since $T$ is a linear transformation, we have

\[
0 = T(u) - T(v) = T(u - v),
\]

so that $u - v \in \text{null}(T)$. Since $\text{null}(T) = \{0\}$, $u - v = 0$ and $u = v$. Thus $T$ is injective.

Range of a Transformation

Another important subset related to a linear transformation is the range:

**Definition 3.17.** The range or image of a linear transformation $T : V \to W$, denoted $\text{range}(T)$, is the set of all $w \in W$ so that there is a $v \in V$ with $T(v) = w$.

**Remark.** The range of a transformation is a subset of the codomain $W$ (unlike the null space, which lives inside the domain $V$).

**Example.** Given any vector spaces $V$ and $W$, the transformation

\[
T_0 : V \to W,
\]

given by

\[
T_0(u) = 0,
\]

has range

\[
\text{range}(T_0) = \{0\},
\]

since every vector in $V$ maps to $0$ under the action of $T$.

**Example.** Let $V$ be any vector space, and let

\[
T_I : V \to V
\]

be defined by

\[
T_I(v) = v.
\]

Every vector in $V$ maps to itself, so every vector is in the range of $T$:

\[
\text{range}(T) = V.
\]
Example. Given linear transformation
\[ \pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R}) \]
defined by
\[ \pi \left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}, \]
find range (\( \pi \)).

The range of \( \pi \) is the set of all vectors in \( \mathcal{U}_2(\mathbb{R}) \) that are images of at least one vector in \( \mathcal{M}_2(\mathbb{R}) \). Of course, it is clear that every vector in \( \mathcal{U}_2(\mathbb{R}) \) is the image of a vector from \( \mathcal{M}_2(\mathbb{R}) \), so
\[ \text{range (} \pi \text{)} = \mathcal{U}_2(\mathbb{R}). \]

Example. With
\[ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \]
find the range of the map \( T_M : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by
\[ T_M \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]

The range of \( T_M \) is the set of all vectors \( w \in \mathbb{R}^3 \) so that there is a solution \( v \in \mathbb{R}^2 \) to the matrix equation
\[ Mv = w. \]
Thus we row reduce the augmented equation for the system:
\[ \begin{pmatrix} 1 & 0 | & w_1 \\ 0 & 1 | & w_2 \\ 1 & 1 | & w_3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 | & w_1 \\ 0 & 1 | & w_2 \\ 0 & 0 | & w_3 - w_1 - w_2 \end{pmatrix}. \]

Notice that there is a solution \( v \in \mathbb{R}^2 \) if and only if
\[ w_3 - w_1 - w_2 = 0; \]
then \( w_1 = v_1, w_2 = v_2 \) and \( w_3 = w_1 + w_2 \). Since \( v_1 \) and \( v_2 \) may be chosen to be any real numbers, we see that the range of \( T_M \) is
\[ \text{range (} T_M \text{)} = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_1 + w_2 \end{pmatrix} \mid w_1, w_2 \in \mathbb{R} \right\}. \]
Again, we have defined this specific subset for a reason: the range of $T$ is a subspace of $W$.

**Theorem 3.19.** If $T : V \to W$ is a linear transformation, then range $(T)$ is a subspace of $W$.

Once again, the proof is quite straightforward: we must show that if $w_1, w_2 \in W$ are vectors in the range of $T$ (i.e., there are vectors $v_1, v_2 \in V$ so that $T(v_1) = w_1, T(v_2) = w_2$), then so is their sum $w_1 + w_2$, and that if $w \in$ range $(T)$ (i.e. there is a vector $v \in V$ with $T(v) = w$), then so is $\lambda w$. Both are easily accomplished using the fact that $T$ is a linear transformation.

Again, we have seen examples of transformations with range $\{0\}$; range $W$; or range somewhere in between. Transformations with range $(T) = W$ are particularly important, and are named below:

**Definition 3.20.** A linear transformation $T : V \to W$ is *onto* or *surjective* if range $(T) = W$.

**Remark.** Surjectivity means that every vector in $W$ is the image of at least one vector in $V$. It is important to distinguish between the ideas of surjectivity and injectivity: they are *not* the same, and we will see momentarily that a map can be surjective but not injective; injective but not surjective; both injective and surjective; or neither injective nor surjective.

**Example.** Given *any* nontrivial vector spaces $V$ and $W$, the transformation

$$T_0 : V \to W$$

given by

$$T_0(u) = 0$$

is neither injective nor surjective.

**Example.** For any $V$, the map

$$T_I : V \to V$$

defined by

$$T_I(v) = v$$

is both injective and surjective.
Example. We showed above that the linear transformation
\[ \pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R}) \]
defined by
\[ \pi \left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix} \]
is surjective, but it is easy to see that it is \textit{not} injective.

Example. With
\[ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} , \]
the map \( T_M : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by
\[ T_M \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]
is injective \textit{but not} surjective.

An Aside on Transformations In Euclidean Space

Of course, the null space and range of a transformation are closely tied to the matrix of the transformation. In order to understand the interconnections, we introduce the idea of the rank of a matrix:

\textbf{Definition.} The \textit{rank} of an \( m \times n \) matrix \( A \), denoted \( \text{rank}(A) \), is the number of leading 1s in rows in its reduced row echelon form.

For example, the matrix
\[ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \]
has reduced row echelon form
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} , \]
so \( \text{rank}(M) = 2 \).

If \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation represented by matrix \( A \), then the rank of \( A \) is closely tied to the dimensions of the null space and range of \( A \). The interconnections are indicated in the following theorem:
Theorem. Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation represented by matrix \( A \) (with respect to any bases). Then:

1. \( \dim(\text{range} \ (T)) = \text{rank} \ (A) \)
2. \( \dim(\text{null} \ (T)) = n - \text{rank} \ (A) \)

We will not consider a proof of the theorem here. Instead, let us briefly compare the result with a problem introduced earlier.

The matrix
\[
M = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{pmatrix}
\]
of the linear transformation \( T_M \) has \( \text{rank} \ (M) = 2 \). In addition, we have seen that \( \dim(\text{range} \ (T_M)) = 2 \) and \( \dim(\text{null} \ (T_M)) = 0 \). Since \( T_M : \mathbb{R}^2 \to \mathbb{R}^3, \ n = 2 \). Notice that

1. \( \dim(\text{range} \ (T_M)) = 2 = \text{rank} \ (M) \)
2. \( \dim(\text{null} \ (T_M)) = 0 = 2 - \text{rank} \ (M), \)

exactly as predicted by the theorem.

Remark. Notice that the theorem suggests that
\[
\dim(\text{range} \ (T)) + \dim(\text{null} \ (T)) = n = \dim(\mathbb{R}^n).
\]
This statement can actually be generalized to apply to any linear transformation between finite dimensional vector spaces, as we will see below.

Fundamental Theorem of Linear Transformations

We sum up all of the data we have accumulated on linear transformations in the following theorem:

**Theorem 3.22: Fundamental Theorem of Linear Transformations.** Let \( T : V \to W \) be a linear transformation. Then
\[
\dim(V) = \dim(\text{null} \ (T)) + \dim(\text{range} \ (T)).
\]

Proof. Let
\[
B_n = (v_1, \ v_2, \ \ldots, \ v_k)
\]
be a basis for null \( (T) \), and extend \( B_n \) to a basis \( B \) for \( V \), say
\[
B = (v_1, \ v_2, \ \ldots, \ v_k, \ v_{k+1}, \ \ldots v_m).
\]
I claim that 

\[ B_r = (T(v_{k+1}), \ldots, T(v_m)) \]

is a basis for range \((T)\). Let \(w\) be any vector in range \((T)\). Then there is some vector \(v \in V\), \(v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m\), with \(T(v) = w\). However, 

\[
T(v) = T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m) \\
= \alpha_1 T(v_1) + \ldots + \alpha_k T(v_k) + \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m) \\
= \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m)
\]

since \(T(v_i) = 0, i \leq k\). Thus 

\[ w = \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m), \]

that is every vector in range \((T)\) is also in span \((B_r)\).

To show that \(B_r\) is a basis for range \((T)\), we must now show that the vectors in \(B_r\) are independent. So suppose that there are constants \(\beta_i\) so that 

\[ \beta_{k+1} T(v_{k+1}) + \ldots + \beta_m T(v_m) = 0. \]

Then the vector \(\beta_{k+1} v_{k+1} + \ldots + \beta_m v_m\) is in the null space of \(T\), and must be a linear combination of vectors in \(B_n\), say 

\[ \beta_1 v_1 + \ldots + \beta_k v_k = \beta_{k+1} v_{k+1} + \ldots + \beta_m v_m. \]

However, the vectors in \(B\) itself are independent, so each \(\beta_i\) in the equation 

\[ \beta_1 v_1 + \ldots + \beta_k v_k - \beta_{k+1} v_{k+1} - \ldots - \beta_m v_m = 0 \]

is identically 0. Thus the vectors \(T(v_{k+1}), \ldots, T(v_m)\) are independent, so that the list \(B_r\) is a basis for range \((T)\).

Now \(\dim(\text{range } (T)) = m - k\) and \(\dim(\text{null } (T)) = k\), so 

\[ \dim(V) = m = \dim(\text{null } (T)) + \dim(\text{range } (T)). \]

We finish the section with a quick corollary:

**Corollaries 3.23/3.24.** Let \(T : V \rightarrow W\) be a linear transformation. Then:

1. If \(\dim V > \dim W\), \(T\) is not injective.
2. If \(\dim V < \dim W\), \(T\) is not surjective.

**Proof.** 1. If \(T\) is injective, then \(\dim(\text{null } (T)) = 0\). Then by the Fundamental Theorem of Linear Maps, we have 

\[
\dim(V) = \dim(\text{null } (T)) + \dim(\text{range } (T)) \\
= \dim(\text{range } (T)).
\]

Now range \((T)\) is a subspace of \(W\), so the equation 

\[ \dim V = \dim(\text{range } (T)) \leq \dim W \]

when \(T\) is injective implies that \(\dim V \leq \dim W\).
2. If $T$ is surjective, then $\text{range } (T) = W$. Again using the Fundamental Theorem, we have

$$\dim V = \dim(\text{null } (T)) + \dim(\text{range } (T))$$

$$= \dim(\text{null } (T)) + \dim W.$$ 

Now $\text{null } (T)$ is a vector space and has dimension $\dim(\text{null } (T)) \geq 0$, so the equation

$$\dim V - \dim W = \dim(\text{null } (T)) \geq 0$$

when $T$ is surjective implies that $\dim V \geq \dim W$. 