# Null Space and Range

As we work to understand linear transformations in more detail, we pause to consider two subspaces that can give us a wealth of information about the transformation itself: the *null space* and the *range*.

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## Null Space of a Transformation

The first subset of interest is the *null space*:

**Definition 3.12.** Let $T : V \to W$ be a linear transformation. The *null space* of $T$, denoted by $\operatorname{null}(T)$, is the set of all vectors $v \in V$ so that $T(v) = \mathbf{0}$.

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**Remark.** Notice that the nullspace is a subset of the domain $V$, and *not* of the codomain $W$. To find $\operatorname{null}(T)$, we must determine which vectors in $V$ are mapped to $\mathbf{0} \in W$.

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**Example.** Given *any* vector spaces $V$ and $W$, the transformation

$$T_{\mathbf{0}} : V \to W,$$

given by

$$T_{\mathbf{0}}(u) = \mathbf{0},$$

has null space

$$\operatorname{null}(T_{\mathbf{0}}) = V,$$

since *every* vector in $V$ maps to $\mathbf{0}$ under the action of $T_{\mathbf{0}}$.

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**Example.** Let $V$ be *any* vector space, and let

$$T_I : V \to V$$

be defined by

$$T_I(v) = v.$$

Then since the *only* vector mapped to $\mathbf{0}$ is $\mathbf{0}$ itself, we see that

$$\operatorname{null}(T_I) = \{\mathbf{0}\}.$$

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**Example.** Given linear transformation

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix},$$

find null $(\pi)$.

The null space of $\pi$ is the set of *all* vectors in $\mathcal{M}_2(\mathbb{R})$ that are mapped to

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{U}_2(\mathbb{R}).$$

If

$$\pi\left( \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we *must* have

$$v_1 = v_2 = v_4 = 0.$$

However, $v_3$ is free, so the null space of $\pi$ is the set

$$\text{null}\,(\pi) = \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \middle| \; v \in \mathbb{R} \right\}.$$

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**Example.** With

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

find the null space of the map $T_M : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T_M\left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Now null $(T_M)$ is the set of all vectors $v \in \mathbb{R}^2$ so that

$$Mv = \mathbf{0}.$$

Of course, to find the set of all solutions to the matrix equation above, we may simply row reduce $M$ itself:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so that there is *only one solution* $v$ to

$$Mv = \mathbf{0}.$$

Thus

$$\text{null}\,(T_M) = \{\mathbf{0}\}.$$

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As you may have already guessed, $\text{null}\,(T)$ is more than just a *subset* of $V$; it is actually a *subspace* of $V$, as indicated by the following theorem:

**Theorem 3.14.** If $T : V \to W$ is a linear transformation, then $\text{null}\,(T)$ is a subspace of $V$.

We will not prove the theorem in detail, as it is quite straightforward. Indeed, the standard procedure applies immediately here: one must show that, if $v$, $w \in \text{null}\,(T)$, then $v + w \in \text{null}\,(T)$ as well (i.e., show that $T(v + w) = \mathbf{0}$ if $T(v) = \mathbf{0}$ and $T(w) = \mathbf{0}$), and that $\lambda v \in \text{null}\,(T)$ (i.e., show that $T(\lambda v) = \mathbf{0}$ if $T(v) = \mathbf{0}$). Both tasks are easily accomplished using the fact that $T$ is a linear transformation.

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We have seen examples of transformations with null space $\{\mathbf{0}\}$; null space $V$; or even null space "in between" $\mathbf{0}$ and $V$. We will soon see that transformations with null space $\text{null}\,(T) = \{\mathbf{0}\}$ are particularly nice. Accordingly, we introduce the following definition:

**Definition 3.15.** A linear transformation $T : V \to W$ is *one-to-one* or *injective* if

$$T(u) = T(v) \to u = v.$$

**Remark.** Injectivity simply means that no two different vectors map to the same place. The idea looks a bit like the concept of invertibility from calculus; while there is a relation, we will see that *invertibility* and *injectivity* are not quite the same, and you should not assume that injectivity implies any sort of "invertibility".

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The idea of an injective map is actually closely tied to the null space of that transformation, as indicated by the following theorem:

**Theorem 3.16.** The linear transformation $T : V \to W$ is injective if and only if $\text{null}\,(T) = \{\mathbf{0}\}$.

**Proof.** For any linear transformation $T$,

$$T(\mathbf{0}) = \mathbf{0}.$$

Thus if $T$ is injective, then *no other* vector $v$ can map to $\mathbf{0}$, so that

$$\text{null}\,(T) = \{\mathbf{0}\}.$$

On the other hand, suppose that $T$ has null space $\{\mathbf{0}\}$, and let $u$ and $v$ be any vectors in $V$ so that $T(u) = T(v)$. Then $T(u) - T(v) = \mathbf{0}$. However, since $T$ is a linear transformation, we have

$$\begin{aligned} \mathbf{0} &= T(u) - T(v) \\ &= T(u - v), \end{aligned}$$

so that $u - v \in \text{null}\,(T)$. Since $\text{null}\,(T) = \{\mathbf{0}\}$, $u - v = \mathbf{0}$ and $u = v$. Thus $T$ is injective.

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## Range of a Transformation

Another important subset related to a linear transformation is the *range*:

**Definition 3.17.** The *range* or *image* of a linear transformation $T : V \to W$, denoted $\text{range}\,(T)$, is the set of all $w \in W$ so that there is a $v \in V$ with $T(v) = w$.

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**Remark.** The range of a transformation is a subset of the codomain $W$ (unlike the null space, which lives inside the domain $V$).

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**Example.** Given *any* vector spaces $V$ and $W$, the transformation

$$T_{\mathbf{0}} : V \to W,$$

given by

$$T_{\mathbf{0}}(u) = \mathbf{0},$$

has range

$$\text{range}\,(T_{\mathbf{0}}) = \{\mathbf{0}\},$$

since *every* vector in $V$ maps to $\mathbf{0}$ under the action of $T$.

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**Example.** Let $V$ be *any* vector space, and let

$$T_I : V \to V$$

be defined by

$$T_I(v) = v.$$

Every vector in $V$ maps to itself, so every vector is in the range of $T$:

$$\text{range}\,(T) = V.$$

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**Example.** Given linear transformation

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left(\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix},$$

find range $(\pi)$.

The range of $\pi$ is the set of *all* vectors in $\mathcal{U}_2(\mathbb{R})$ that are images of at least one vector in $\mathcal{M}_2(\mathbb{R})$. Of course, it is clear that *every* vector in $\mathcal{U}_2(\mathbb{R})$ is the image of a vector from $\mathcal{M}_2(\mathbb{R})$, so

$$\text{range}\,(\pi) = \mathcal{U}_2(\mathbb{R}).$$

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**Example.** With

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

find the range of the map $T_M : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The range of $T_M$ is the set of all vectors $w \in \mathbb{R}^3$ so that there is a solution $v \in \mathbb{R}^2$ to the matrix equation

$$Mv = w.$$

Thus we row reduce the augmented equation for the system:

$$\left(\begin{array}{cc|c} 1 & 0 & w_1 \\ 0 & 1 & w_2 \\ 1 & 1 & w_3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & w_1 \\ 0 & 1 & w_2 \\ 0 & 0 & w_3 - w_1 - w_2 \end{array}\right).$$

Notice that there is a solution $v \in \mathbb{R}^2$ if and only if

$$w_3 - w_1 - w_2 = 0;$$

then $w_1 = v_1$, $w_2 = v_2$ and $w_3 = w_1 + w_2$. Since $v_1$ and $v_2$ may be chosen to be any real numbers, we see that the range of $T_M$ is

$$\text{range}\,(T_M) = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_1 + w_2 \end{pmatrix} \middle| w_1,\ w_2 \in \mathbb{R} \right\}.$$

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Again, we have defined this specific subset for a reason: the range of $T$ is a subspace of $W$.

**Theorem 3.19.** If $T : V \to W$ is a linear transformation, then $\text{range}\,(T)$ is a subspace of $W$.

Once again, the proof is quite straightforward: we must show that if $w_1$, $w_2 \in W$ are vectors in the range of $T$ (i.e., there are vectors $v_1$, $v_2 \in V$ so that $T(v_1) = w_1$, $T(v_2) = w_2$), then so is their sum $w_1 + w_2$, and that if $w \in \text{range}\,(T)$ (i.e. there is a vector $v \in V$ with $T(v) = w$), then so is $\lambda w$. Both are easily accomplished using the fact that $T$ is a linear transformation.

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Again, we have seen examples of transformations with range $\{\mathbf{0}\}$; range $W$; or range somewhere in between. Transformations with $\text{range}\,(T) = W$ are particularly important, and are named below:

**Definition 3.20.** A linear transformation $T : V \to W$ is *onto* or *surjective* if $\text{range}\,(T) = W$.

**Remark.** Surjectivity means that every vector in $W$ is the image of at least one vector in $V$. It is important to distinguish between the ideas of surjectivity and injectivity: they are *not* the same, and we will see momentarily that a map can be surjective but not injective; injective but not surjective; both injective and surjective; or neither injective nor surjective.

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**Example.** Given *any* nontrivial vector spaces $V$ and $W$, the transformation

$$T_{\mathbf{0}} : V \to W$$

given by

$$T_{\mathbf{0}}(u) = \mathbf{0}$$

is neither injective nor surjective.

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**Example.** For any $V$, the map

$$T_I : V \to V$$

defined by

$$T_I(v) = v$$

is both injective and surjective.

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**Example.** We showed above that the linear transformation

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left(\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}$$

is surjective, but it is easy to see that it is *not* injective.

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**Example.** With

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

the map $T_M : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is injective *but not* surjective.

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## An Aside on Transformations In Euclidean Space

Of course, the null space and range of a transformation are closely tied to the matrix of the transformation. In order to understand the interconnections, we introduce the idea of the *rank* of a matrix:

**Definition.** The *rank* of an $m \times n$ matrix $A$, denoted $\operatorname{rank}(A)$, is the number of leading 1s in rows in its reduced row echelon form.

For example, the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

has reduced row echelon form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so $\operatorname{rank}(M) = 2$.

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If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation represented by matrix $A$, then the rank of $A$ is closely tied to the dimensions of the null space and range of $A$. The interconnections are indicated in the following theorem:

**Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation represented by matrix $A$ (with respect to any bases). Then:

1. $\dim(\text{range}\,(T)) = \text{rank}\,(A)$
2. $\dim(\text{null}\,(T)) = n - \text{rank}\,(A)$

We will not consider a proof of the theorem here. Instead, let us briefly compare the result with a problem introduced earlier.

The matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

of the linear transformation $T_M$ has $\text{rank}\,(M) = 2$. In addition, we have seen that $\dim(\text{range}\,(T_M)) = 2$ and $\dim(\text{null}\,(T_M)) = 0$. Since $T_M : \mathbb{R}^2 \to \mathbb{R}^3$, $n = 2$. Notice that

1. $\dim(\text{range}\,(T_M)) = 2 = \text{rank}\,(M)$
2. $\dim(\text{null}\,(T_M)) = 0 = 2 - \text{rank}\,(M)$,

exactly as predicted by the theorem.

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**Remark.** Notice that the theorem suggests that

$$\dim(\text{range}\,(T)) + \dim(\text{null}\,(T)) = n = \dim(\mathbb{R}^n).$$

This statement can actually be generalized to apply to *any* linear transformation between finite dimensional vector spaces, as we will see below.

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## Fundamental Theorem of Linear Transformations

We sum up all of the data we have accumulated on linear transformations in the following theorem:

**Theorem 3.22: Fundamental Theorem of Linear Transformations.** Let $T : V \to W$ be a linear transformation. Then

$$\dim(V) = \dim(\text{null}\,(T)) + \dim(\text{range}\,(T)).$$

**Proof.** Let

$$B_n = (v_1,\ v_2,\ \ldots,\ v_k)$$

be a basis for $\text{null}\,(T)$, and extend $B_n$ to a basis $B$ for $V$, say

$$B = (v_1,\ v_2,\ \ldots,\ v_k,\ v_{k+1},\ \ldots v_m).$$

I claim that

$$B_r = (T(v_{k+1}),\ \ldots,\ T(v_m)))$$

is a basis for range $(T)$. Let $w$ be any vector in range $(T)$. Then there is some vector $v \in V$, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_m$, with $T(v) = w$. However,

$$\begin{aligned} T(v) &= T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m) \\ &= \alpha_1 T(v_1) + \ldots + \alpha_k T(v_k) + \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m) \\ &= \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m) \end{aligned}$$

since $T(v_i) = \mathbf{0}$, $i \leq k$. Thus

$$w = \alpha_{k+1} T(v_{k+1}) + \ldots + \alpha_m T(v_m),$$

that is every vector in range $(T)$ is also in span $(B_r)$.

To show that $B_r$ is a basis for range $(T)$, we must now show that the vectors in $B_r$ are independent. So suppose that there are constants $\beta_i$ so that

$$\beta_{k+1} T(v_{k+1}) + \ldots + \beta_m T(v_m) = \mathbf{0}.$$

Then the vector $\beta_{k+1} v_{k+1} + \ldots + \beta_m v_m$ is in the null space of $T$, and must be a linear combination of vectors in $B_n$, say

$$\beta_1 v_1 + \ldots + \beta_k v_k = \beta_{k+1} v_{k+1} + \ldots + \beta_m v_m.$$

However, the vectors in $B$ itself are independent, so each $\beta_i$ in the equation

$$\beta_1 v_1 + \ldots + \beta_k v_k - \beta_{k+1} v_{k+1} - \ldots - \beta_m v_m = \mathbf{0}$$

is identically 0. Thus the vectors $T(v_{k+1}), \ldots, T(v_m)$ are independent, so that the list $B_r$ is a basis for range $(T)$.

Now $\dim(\text{range}\,(T)) = m - k$ and $\dim(\text{null}\,(T)) = k$, so

$$\dim(V) = m = \dim(\text{null}\,(T)) + \dim(\text{range}\,(T)).$$

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We finish the section with a quick corollary:

**Corollaries 3.23/3.24.** Let $T : V \to W$ be a linear transformation. Then:

1. If $\dim V > \dim W$, $T$ is not injective.
2. If $\dim V < \dim W$, $T$ is not surjective.

**Proof.** 1. If $T$ is injective, then $\dim(\text{null}\,(T)) = 0$. Then by the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} \dim(V) &= \dim(\text{null}\,(T)) + \dim(\text{range}\,(T)) \\ &= \dim(\text{range}\,(T)). \end{aligned}$$

Now range $(T)$ is a subspace of $W$, so the equation

$$\dim V = \dim(\text{range}\,(T)) \leq \dim W$$

when $T$ is injective implies that $\dim V \leq \dim W$.

2. If $T$ is surjective, then $\text{range}\,(T) = W$. Again using the Fundamental Theorem, we have

$$\begin{aligned} \dim V &= \dim(\text{null}\,(T)) + \dim(\text{range}\,(T)) \\ &= \dim(\text{null}\,(T)) + \dim W. \end{aligned}$$

Now $\text{null}\,(T)$ is a vector space and has dimension $\dim(\text{null}\,(T)) \geq 0$, so the equation

$$\dim V - \dim W = \dim(\text{null}\,(T)) \geq 0$$

when $T$ is surjective implies that $\dim V \geq \dim W$.