Linear Transformations of Vector Spaces

As we begin to think more deeply about the structure of vector spaces, our next logical step is to understand vector spaces in relation with each other–indeed, to introduce maps or functions between the spaces.

Of course, we would like our maps to preserve the key data about a vector space: the way that vector addition and scalar multiplication work. Thus we introduce the following definition:

Definition 3.2. Let V and W be vector spaces over $\mathbb{F}$. A *linear transformation* $T : V \to W$ is a function from V to W satisfying the following properties:

1. $T(u + v) = T(u) + T(v)$ for all u, $v \in V$, and
2. $T(\lambda u) = \lambda T(u)$ for all $u \in V$, $\lambda \in \mathbb{F}$.

The definition above simply requires that the map T in question interact nicely with the vector space operations.

Example. The 0 transformation

Given *any* vector spaces V and W, the transformation

$$T_{\mathbf{0}} : V \to W,$$

given by

$$T_{\mathbf{0}}(u) = \mathbf{0},$$

(*every* vector in V gets sent to $\mathbf{0} \in W$), is a linear transformation.

Example. The identity transformation

Let V be *any* vector space, and let

$$T_I : V \to V$$

be defined by

$$T_I(v) = v,$$

that is, T_I sends every vector to itself; T_I is called the *identity transformation.* Then clearly T_I is a linear transformation.

Example. The function

$$\iota : \mathcal{U}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$$

defined by

$$\iota\left(\begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}\right) = \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}$$

is a linear transformation. Notice that ι is *not* the identity transformation, since its domain and codomain are different; ι is often referred to as an *embedding.*

Example. The function

$$\pi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$\pi\left(\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \right) = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 \end{pmatrix}$$

is a linear transformation. The transformation π is called a *projection*.

Example. Show that the function

$$T : \mathbb{R}^3 \to \mathcal{U}_2(\mathbb{R})$$

defined by

$$T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) = \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}$$

is a linear transformation.

To be convinced that T is a transformation, we must show that

1. $T(u + v) = T(u) + T(v)$ for all u, $v \in \mathbb{R}^3$, and
2. $T(\lambda u) = \lambda T(u)$ for all $u \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$.

Let's begin by showing that T interacts nicely with addition:

$$\begin{aligned} T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) &= T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} u_1 + v_1 & u_2 + v_2 \\ 0 & u_3 + v_3 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \\ &= T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) + T\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right). \end{aligned}$$

Thus T preserves addition.

Next, we check that T preserves scalar multiplication:

$$\begin{aligned} T\left(\lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) &= T\left(\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}\right) \\ &= \begin{pmatrix} \lambda u_1 & \lambda u_2 \\ 0 & \lambda u_3 \end{pmatrix} \\ &= \lambda \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix} \\ &= \lambda T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right). \end{aligned}$$

Again, T interacts nicely with the operation. Thus T is indeed a linear transformation.

Key Point. If V and W are different spaces, then they have different addition and scalar multiplication operations. Thus a statement such as

$$T(u+v) = T(u) + T(v)$$

is a bit ambiguous–the first "+" refers to addition in V, whereas the second refers to addition in W. However, a linear transformation draws parallels between the two operations, indicating that they are in some sense analogous; T preserves the operations even as it changes the ambient spaces.

Example. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} u_1 \\ u_1 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix}$$

is a linear transformation.

To show that T is actually a linear transformation, we need to be show that it interacts nicely with the addition and scalar multiplication operations. First, we check that $T(u+v) = T(u)+T(v)$:

$$\begin{aligned}
T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) &= T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \right) \\
&= \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_1 + u_2 + v_1 + v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_1 + v_2 \end{pmatrix} \\
&= T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) + T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right).
\end{aligned}$$

Next, we check that $T(\lambda u) = \lambda T(u)$:

$$\begin{aligned}
T\left(\lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) &= T\left(\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} \right) \\
&= \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_1 + \lambda u_2 \end{pmatrix} \\
&= \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix} \\
&= \lambda T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right).
\end{aligned}$$

T passes both conditions, and is thus a linear transformation.

Example. Set

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that the map T_M, defined by

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

(that is, T_M acts on vectors in $\mathbb{R}^2$ via matrix multiplication by M) is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

It is actually quite easy to see that T_M is a linear transformation by using properties of matrix arithmetic. First of all, since M is 3×1, $T_M(u) = Mu$ will be a 3×1 matrix for all $u \in \mathbb{R}^2$, that is

$$T_M(u) \in \mathbb{R}^3.$$

Second,

$$\begin{aligned} T_M(u+v) &= M(u+v) \\ &= Mu + Mv \\ &= T_M(u) + T_M(v) \end{aligned}$$

since matrix multiplication distributes over matrix addition.

Finally,

$$\begin{aligned} T_M(\lambda u) &= M(\lambda u) \\ &= \lambda M u \\ &= \lambda T_M(u), \end{aligned}$$

again using properties of matrix arithmetic.

Thus the map T_M is a linear transformation.

Example. Given linear transformations T and T_M from the last two examples, show that

$$T(u) = T_M(u)$$

for all $u \in \mathbb{R}^2$.

Let's begin by calculating the form of the vector $T_M(u) \in \mathbb{R}^3$:

$$\begin{aligned} T_M(u) &= T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + 0 \\ 0 + u_2 \\ u_1 + u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix}. \end{aligned}$$

Now the resulting matrix is precisely the image of the original vector u under T, that is

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right),$$

so that

$$T_M = T.$$

Remark. We will see that the last example is extremely important, and indeed indicative of the interconnections between matrices and linear transformations. In the next section of this unit, we will discuss these interconnections in detail.

Bases and Linear Maps

Since a basis for a vector space tells us virtually everything we need to know about the space, it seems reasonable to guess that a chosen basis for V can be thought of as the controlling factor for linear transformations out of V. The following theorem confirms that intuition:

Theorem 3.5. Let $v_1, v_2, \ldots, v_n$ be a basis for V, and let $w_1, w_2, \ldots, w_n$ be *any* list of vectors in W. Then there is a unique linear transformation $T : V \to W$ so that $T(v_i) = w_i$ for all i, $1 \leq i \leq n$.

Proof. Define a map T on the *basis* $(v_1,\ v_2,\ \ldots,\ v_n)$ by $T(v_i) = w_i$, $1 \leq i \leq n$; we will extend the map linearly to all of V.

Given $u \in V$ written with respect to this basis as

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n,$$

define $T(u)$ using the rule

$$T(u) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n).$$

Now every $u \in V$ is a linear combination of vectors in the list $(v_1,\ v_2,\ \ldots,\ v_n)$, and every linear combination of $T(v_1)$, $T(v_2)$, $\ldots$, $T(v_n)$ must be a vector in W, so clearly the T may be applied to any vector in V, and the resulting vector is in W; that is,

$$T : V \to W.$$

It remains to show that T is a linear transformation.

Given u, $v \in V$ with

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \text{ and } v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n,$$

the definition for T implies that

$$\begin{aligned}
T(u+v) &= T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n) \\
&= T((\alpha_1+\beta_1)v_1 + (\alpha_2+\beta_2)v_2 + \ldots + (\alpha_n+\beta_n)v_n) \\
&= (\alpha_1+\beta_1)T(v_1) + (\alpha_2+\beta_2)T(v_2) + \ldots + (\alpha_n+\beta_n)T(v_n) \\
&= \alpha_1 T(v_1) + \beta_1 T(v_1) + \alpha_2 T(v_2) + \beta_2 T(v_2) + \ldots + \alpha_n T(v_n) + \beta_n T(v_n) \\
&= (\alpha_1 T(v_1) + \alpha_1 T(v_2) + \ldots + \alpha_1 T(v_n)) + (\beta_1 T(v_1) + \beta_1 T(v_2) + \ldots + \beta_1 T(v_n)) \\
&= T(u) + T(v).
\end{aligned}$$

Thus T satisfies the linearity property on addition. Next, we check that T is linear on scalar multiplication.

Given $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ and $\lambda \in \mathbb{F}$,

$$\begin{aligned}
T(\lambda u) &= T(\lambda(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)) \\
&= T(\lambda\alpha_1 v_1 + \lambda\alpha_2 v_2 + \ldots + \lambda\alpha_n v_n) \\
&= \lambda\alpha_1 T(v_1) + \lambda\alpha_2 T(v_2) + \ldots + \lambda\alpha_n T(v_n) \\
&= \lambda(\alpha_1 T(v_1) + \alpha_1 T(v_2) + \ldots + \alpha_1 T(v_n)) \\
&= \lambda T(u).
\end{aligned}$$

Thus T is linear over both scalar multiplication *and* vector addition on V, and is a linear transformation.

Finally, we need to check that T is unique; so suppose that there is another linear transformation $S : V \to W$ so that

$$S(v_i) = w_i$$

for all i. Now S is a *linear transformation*, so for any $v \in V$,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n,$$

we have

$$\begin{aligned}
S(v) &= S(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) \\
&= S(\alpha_1 v_1) + S(\alpha_2 v_2) + \ldots + S(\alpha_n v_n) \\
&= \alpha_1 S(v_1) + \alpha_2 S(v_2) + \ldots + \alpha_n S(v_n) \\
&= \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \\
&= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n) \\
&= T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) \\
&= T(v).
\end{aligned}$$

So $T(v) = S(v)$ for all $v \in V$, so that $T = S$. Thus T is unique.

Key Point. A linear transformation is *completely determined by its action on a basis.*

Example. Given

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and the associated linear transformation T_M defined by

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

determine the action of T_M on the standard basis for $\mathbb{R}^2$.

The standard basis vectors e_1 and e_2 for $\mathbb{R}^2$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Applying T_M to these vectors, we see that

$$\begin{aligned} T_M(e_1) &= Me_1 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} T_M(e_2) &= Me_2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now the theorem tells us that T_M is completely determined by its action on e_1 and e_2; so if we like, we may rewrite our definition of T_M to see this more clearly. Since any vector u in $\mathbb{R}^2$ may be written as

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1e_1 + u_2e_2,$$

where u_1, $u_2 \in \mathbb{R}$, we see that we can think of T_M as the linear transformation so that

$$\begin{aligned} T_M(u) &= T_M(u_1e_1 + u_2e_2) \\ &= u_1T_M(e_1) + u_2T_M(e_2) \\ &= u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix}. \end{aligned}$$

This last equation should look familiar–indeed, we saw earlier in this section that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} \right) = \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix}$$

is exactly T_M, that is $T = T_M$; we have just confirmed this equality again.

The Vector Space of Linear Transformations

Definition 3.3. The set of all linear transformations from V to W is denoted by $\mathcal{L}(V, W)$.

We will see momentarily that the set $\mathcal{L}(V, W)$ of linear transformations from V to W has a great deal of structure inherited from the structure of the background spaces; perhaps unsurprisingly, $\mathcal{L}(V, W)$ is itself a vector space over the same field.

Of course, in order to be a vector space, $\mathcal{L}(V, W)$ must have two operations–thus we take a moment to define vector addition and scalar multiplication on $\mathcal{L}(V, W)$.

Definition 3.6. Given S, $T \in \mathcal{L}(V, W)$, (that is S and T are both linear transformations from V to W), the *sum* $S + T$ of S and T is the linear transformation defined by the rule

$$(S + T)(v) = S(v) + T(v).$$

If V and W are vector spaces over $\mathbb{F}$, then the *scalar product* λS of $S \in \mathcal{L}(V, W)$ with $\lambda \in \mathbb{F}$ is the linear transformation defined by the rule

$$(\lambda S)(v) = \lambda \cdot S(v).$$

Notice that the definition above *claims* that maps $S + T$ and λS are linear transformations if S and T are; technically we should prove this assertion, but the proof is standard enough that we will leave it as an exercise.

Theorem 3.7. The set $\mathcal{L}(V, W)$ of all linear transformations from V to W with the operations of vector addition and scalar multiplication as defined in Definition 3.6 is a vector space.

Again, the proof is quite straightforward, and we leave it as an exercise.

Products of Linear Transformations

In general, a vector space has only two operations defined on it; however, it is possible to define an extra operation for certain linear transformations.

Definition 3.8. Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. Then the product ST defined by

$$ST(u) = S(T(u))$$

is a linear transformation, and

$$ST : U \to W,$$

that is $ST \in \mathcal{L}(U, W)$.

There are several important points to notice about the definition:

1. The definition asserts that ST is a linear transformation. Technically we should check this statement, but once more this is quite routine, and is left as an exercise.

2. In order to define the "product" of a pair of linear transformations, we used function composition; indeed, the product ST acts on u by first applying T to u, then applying S to the resulting vector.

3. The product of a pair of transformations is *only* defined if the "middle spaces" match up (this may make you think of the definition of matrix multiplication, which is only defined if the number of columns of the first matrix matches up with the number of rows of the second.)

4. The order of the transformations is important. With $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, suppose that we wish to apply the product to $u \in U$. Now we *must* apply T first, since $T : U \to V$; thus we write $T(u)$. Next, since $T(u) \in V$, we may now apply S, and write $S(T(u))$ or $ST(u)$, which is an element of W. In particular, it is easy to think that we should write "TS" for the product of the transformations, but given the way that we denote function composition, TS implies that we *first* apply S, *then* apply T (which would not make sense). In particular, with $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, the product TS *is not defined* if $U \neq W$.

Example. Consider

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and its associated linear transformation T_M, defined by

$$T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Let

$$N = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Then it is easy to see that the map $T_N : \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$T_N\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) = N\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

is a linear transformation. Compute the linear transformation defined by the product

$$T_N T_M.$$

Notice that, since

$$T_M \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \text{ and } T_N \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^1),$$

the product

$$T_N T_M \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^1).$$

Of course, this makes sense since we will apply the 3×2 matrix M to a 2×1 matrix, resulting in a 3×1 matrix; multiplying afterwards by the 1×3 N results in a 1×1 matrix.

Let's make the calculation:

$$\begin{aligned}
T_N T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= T_N\left(T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right)\right) \\
&= T_N\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \\
&= T_N\left(\begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 + u_1 + u_2 \end{pmatrix} \\
&= \begin{pmatrix} 2u_1 + 2u_2 \end{pmatrix}.
\end{aligned}$$

Thus the linear transformation $T_N T_M \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^1)$ is defined by

$$T_N T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} 2u_1 + 2u_2 \end{pmatrix}.$$

Example. Show that

$$T_N T_M(u) = (NM)u.$$

Since

$$T_N T_M(u) = N(Mu) = (NM)u$$

(using properties of matrix arithmetic), we note that the product $T_N T_M$ is really defined by multiplying $u \in \mathbb{R}^2$ by the matrix NM. Since

$$\begin{aligned} NM &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \end{pmatrix}, \end{aligned}$$

then

$$T_N T_M\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

We record without proof a few observations on the properties of the product of transformations:

Theorem 3.9.

1. If T_1, T_2, and T_3 are linear transformations so that the products

   $$T_1 T_2 \text{ and } T_2 T_3$$

   make sense, then

   $$(T_1 T_2)T_3 = T_1(T_2 T_3)$$

   (associativity).

2. For $T \in \mathcal{L}(V, W)$ and T_I the identity transformation on V,

   $$TT_I = T;$$

   similarly, if T_I is the identity transformation on W, then

   $$T_I T = T$$

   (T_I acts like a multiplicative identity).

3. If T, T_1, $T_2 \in \mathcal{L}(U, V)$ and S, S_1, $S_2 \in \mathcal{L}(V, W)$, then

   $$S(T_1 + T_2) = ST_1 + ST_2 \text{ and } (S_1 + S_2)T = S_1 T + S_2 T$$

   (distribution of multiplication over addition).

Finally, we record an observation about the action of a transformation on the $\mathbf{0}$ vector:

Theorem 3.11. If $T \in \mathcal{L}(U, V)$, then

$$T(\mathbf{0}_U) = \mathbf{0}_V,$$

where the notation $\mathbf{0}_U$ indicates the zero vector in U.