The Dimension of a Vector Space

We have spent a great deal of time and effort on understanding the geometry of vector spaces, but we have not yet discussed an important geometric idea–that of the *size* of the space.

For example, think about the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Which one is "bigger"? We have not defined precisely what we mean by "bigger" or "smaller", but intuitively, you know that $\mathbb{R}^3$ is the larger space.

Now, the set $\mathcal{M}_2(\mathbb{R})$ of all 2×2 real matrices is also a vector space, so we could play the same game with it: which is the bigger vector space, $\mathcal{M}_2(\mathbb{R})$ or $\mathbb{R}^2$? This time, intuition fails us, because we are not used to thinking of $\mathcal{M}_2(\mathbb{R})$ spatially.

In this section, we will define precisely what we mean by the "size" of a vector space, thereby giving us the tools to answer such questions. In addition, we will see how the size of a vector space is closely related to linear independence and spanning.

Basis Size and Dimension

In the previous section, we saw that the set

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$. We also saw that the set

$$B_2 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^2$ as well.

It is interesting to note that both of the bases that we have found for $\mathbb{R}^2$ have *two* elements. Of course, there are *many* other bases for $\mathbb{R}^2$; is it possible to find a basis containing, say 3 vectors? Or a basis with only 1 vector? The following theorem answers this question in a surprising and mathematically beautiful way:

Theorem 2.35. If the finite dimensional vector space V has a basis consisting of n elements, then *every* basis for V has n elements.

Theorem 2.35 answers the question above in the negative: since we know of a basis for $\mathbb{R}^2$ that has two elements, then *any* basis that we can find for $\mathbb{R}^2$ has to have 2 elements. Similarly, we know that the set $\mathcal{M}_2(\mathbb{R})$ of all real 2×2 matrices has a basis consisting of the matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

the theorem says that *any* other basis you can find for $\mathcal{M}_2(\mathbb{R})$ will also have 4 elements.

Proof of Theorem 2.35. Let S and S' be bases for vector space V. Theorem 2.23 states that the length of an independent list is no more than the length of a spanning list. S is independent and S' is spanning, so

$$|S| \leq |S'|.$$

Reversing the roles of S, and S', we see that

$$|S'| \leq |S|.$$

Thus

$$|S| = |S'|,$$

so that every basis for V has the same number of elements.

The theorem now gives us an unambiguous way to define what we mean when we refer to the size of a vector space:

Definition 2.36. The *dimension* of a finite dimensional vector space V, denoted $\dim(V)$, is the number of vectors in a basis for V. We define the dimension of the trivial vector space (i.e., the vector space containing only $\mathbf{0}$) to be 0.

In a sense, the dimension of a vector space tells us how many vectors are needed to "build" the space, thus gives us a way to compare the relative sizes of the spaces. Using the theorem, it is clear that

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathfrak{sl}(2, \mathbb{R})) = 3$
- $\dim(\mathcal{M}_{mn}(\mathbb{F})) = mn$
- $\dim(\mathcal{P}_n(\mathbb{F})) = n + 1$
- $\dim(\mathcal{U}_2(\mathbb{R})) = 3$.

In particular, our original observation that $\mathbb{R}^3$ is a "larger" space than is $\mathbb{R}^2$ is correct (and now defined more precisely). Similarly, we can conclude that $\mathcal{M}_2(\mathbb{R})$ is a larger space that $\mathbb{R}^3$, as it takes more vectors to build the space.

With the previous theorem in mind, we take a moment to consider the dimensions of subspaces. Recall that a *subspace* of a vector space V is a subset U of the vectors of V, which is a vector space in its own right (with the same operations as V).

For example, the vector space $\mathcal{U}_2(\mathbb{R})$ is a subspace of $\mathcal{M}_2(\mathbb{R})$. Of course, we expect $\mathcal{U}_2(\mathbb{R})$ to be relatively small compared to the ambient space, thus to have smaller dimension (as confirmed by the list above). The following theorem confirms this observation in general:

Theorem 2.38. If U is a subspace of a finite dimensional vector space V, then

$$\dim(U) \leq \dim(V).$$

Proof. A basis S for U consists of linearly independent vectors in V, and a basis S' for V is a spanning list. Recalling Theorem 2.23, which states that the length of an independent list in V is never more than the length of a spanning list in V, we see that

$$|S| \leq |S'|.$$

Understanding Bases

At this point, it should be clear that a basis for a vector space V has to have *enough* vectors to be able to span V, but not so many that it is no longer linearly independent. We make this idea precise with the following theorem:

Theorem. Let V be an n-dimensional vector space, that is, every basis of V consists of n vectors. Then

(a) Any list of vectors from V containing *more than* n vectors is linearly dependent.

(b) Any list of vectors from V containing *fewer than* n vectors does not span V.

The theorem is really just a restatement (in terms of bases) of a result that we already know: for part (a), we recall that the length of an independent list is no longer than the length of a spanning list, so clearly a list that is longer than the length of a list of basis vectors must be dependent. Similarly for part (b), a list that is shorter than the length of a basis cannot be spanning.

Key Point. Adding too many vectors to a set will force the set to be linearly dependent; on the other hand, taking too many vectors away from a set will prevent it from spanning. A basis is a sort of "sweet spot" between linear independence and spanning: if a basis S for V has n elements and we remove one, then S no longer spans; add one, and S is no longer linearly independent.

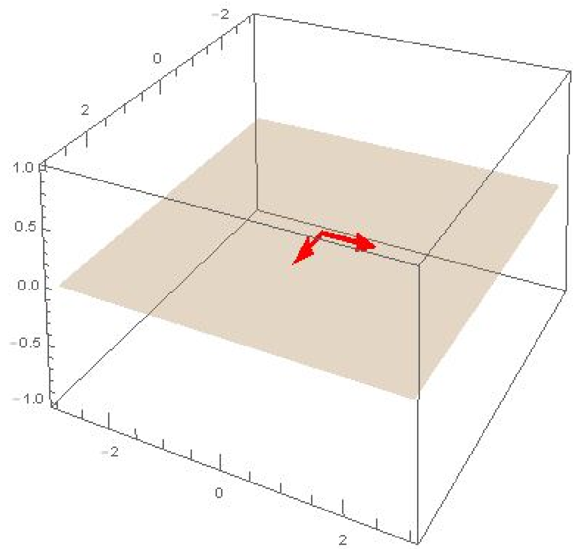
As an example, we know that the list

$$B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis for $\mathbb{R}^3$; it is linearly independent, and it spans $\mathbb{R}^3$. If we remove the last vector, to get the set

$$B' = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

then we no longer have a spanning list for $\mathbb{R}^3$. Indeed, this is easy to see geometrically: the two vectors from B' are graphed below in $\mathbb{R}^3$, and clearly cannot be used to describe the vertical component of the space:



In fact, these two vectors span the *subspace* of $\mathbb{R}^3$ shaded in gray, but not all of $\mathbb{R}^3$.

The theorems above lead to some important facts about the geometry of vector spaces and their subspaces.

Theorems 2.39/2.42. Let V be an n-dimensional vector space.

1. Any independent list of n vectors in V is a basis.
2. Any spanning list of n vectors in V is a basis.

Key Point. We know that a basis for a vector space must be a linearly independent spanning set; to conclude that a set is a basis, we must be certain that both conditions are met.

However, Theorems 2.39 and 2.42 make it much easier to determine whether or not a set is a basis: if a set has the right number of vectors–the same as the dimension of V–then we can quickly check to see if the set is a basis by determining if it is a linearly independent set, or alternatively by checking that the set spans V. We don't have to check *both* conditions anymore, just one of them!

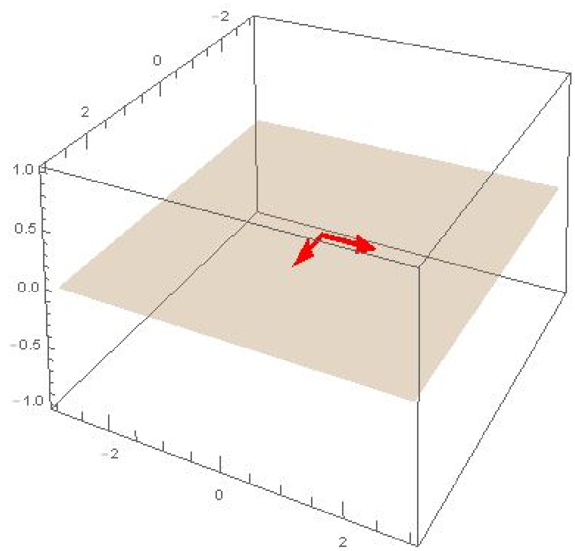
Proof of Theorems 2.39/2.42.

1. Suppose that V is an n-dimensional vector space, and let S be a list of n linearly independent vectors in V. If S is not a basis, then it may be extended to a basis S' by Theorem 2.33. However, every basis of V has n vectors, so the extension to S' was trivial, so that S must be a basis for V.
2. Let S be a spanning list consisting of n vectors. If S is not a basis, then it must contain a basis by Theorem 2.31. However, any sublist of S obtained nontrivially has fewer than n vectors and cannot be a basis. Thus S must already be a basis for V.

Let's use the theorems to investigate the example introduced above: the set

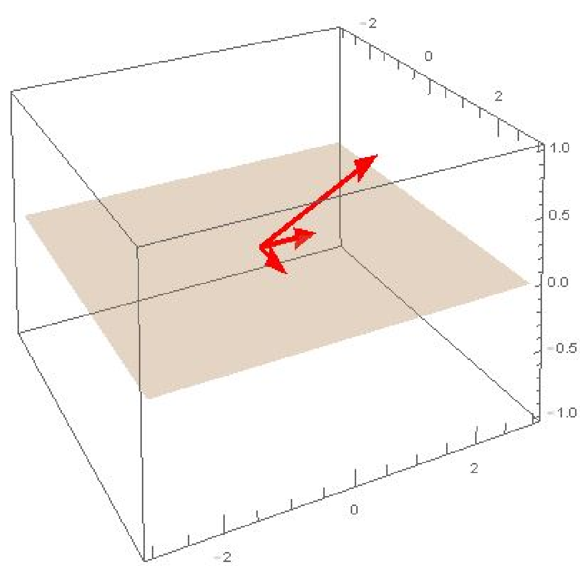
$$B' = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

while linearly independent in $\mathbb{R}^3$, *does not* span $\mathbb{R}^3$:



However, if we extend B' by adding *any* linearly independent vector, we will have a list of the right length, and thus a basis. In other words, we can add *any* vector we like to B' (as long as that vector is not already in the span of B'), and we will then have a basis. In this case, we just need to choose a vector that does not lie in the xy plane–for example, we could choose

$$\left(2, 1, 1\right):$$



It is easy to check that the list

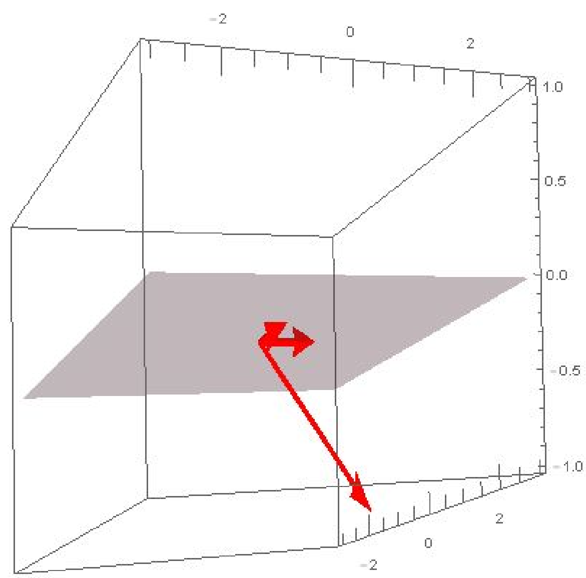
$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right)$$

is linearly independent, and thus a basis for $\mathbb{R}^3$, as $\mathbb{R}^3$ is 3 dimensional. Since the set is independent and has the right number of vectors, Theorem 2.39 tells us that we don't have to check that it spans $\mathbb{R}^3$ to know that it's a basis!

Alternatively, we could choose the vector

$$\big(0, 3, -1\big)$$

to add to B':



Again, it is easy to check that the list

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\-1\end{pmatrix} \right)$$

is linearly independent, so the theorem indicates that it is also a basis.

Example

The vectors

$$f_1(x) = 2x - 3,\ f_2(x) = x^2 + 1,\ \text{and}\ f_3(x) = 2x^2 - x$$

are linearly independent. Complete the list to form a basis for $\mathcal{P}_3(\mathbb{R})$, the space of all real-valued polynomials of degree no more than 3.

We know that a basis for $\mathcal{P}_3(\mathbb{R})$ must consist of 4 linearly independent vectors that span $\mathcal{P}_3(\mathbb{R})$. Since we already have 3 linearly independent vectors, we simply need to find one more independent vector to add to the list.

It is not too difficult to find a vector independent from those in our current list–since none of the polynomials in our list include a term of x^3, $f_4(x) = x^3$ seems like a good choice. Indeed, it is clear that x^3 is *not* a linear combination of f_1, f_2, and f_3, so the list

$$(2x - 3,\ x^2 + 1,\ 2x^2 - x,\ x^3)$$

of four independent vectors in the 4-dimensional vector space $\mathcal{P}_3(\mathbb{R})$ is a basis.

Dimensions of Sums and Direct Sums

Finally, we record a theorem that will allow us to quickly calculate the dimension of a sum of subspaces.

Theorem 2.43. If U_1 and U_2 are subspaces of a vector space V, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Remark. Recall from a previous homework assignment that the intersection of a pair of subspaces is also a subspace (and thus a vector space). Thus the notation $\dim(U_1 \cap U_2)$ is justified.

Proof. Let

$$S' = (u_1,\ u_2,\ \ldots,\ u_k)$$

be a basis for $U_1 \cap U_2$. Of course, S' is independent, so by Theorem 2.33, we may extend S' to bases

$$S_1 = (u_1,\ u_2,\ \ldots,\ u_k,\ u_{11},\ \ldots,\ u_{1m})$$

and

$$S_2 = (u_1,\ u_2,\ \ldots,\ u_k,\ u_{21},\ \ldots,\ u_{2n})$$

for U_1 and U_2, respectively. Set

$$M = S_2 - S_1 \cap S_2 = (u_{21},\ \ldots,\ u_{2n}).$$

Now I claim that the list

$$S = S_1 \cup M = (u_1,\ u_2,\ \ldots,\ u_k,\ u_{11},\ \ldots,\ u_{1m},\ u_{21},\ \ldots,\ u_{2n})$$

is a basis for $U_1 + U_2$. First, let us prove that S spans $U_1 + U_2$: if $v \in U_1 + U_2$, then v may be written as

$$v = u_1' + u_2',$$

where $u_1' \in U_1$ and $u_2' \in U_2$. Now u_1' is a linear combination of vectors in S_1, say

$$u_1' = \alpha_1 u_1 + \ldots + \alpha_k u_k + \alpha_{11} u_{11} + \ldots + \alpha_{1m} u_{1m},$$

and similarly

$$u_2' = \beta_1 u_1 + \ldots + \beta_k u_k + \beta_{21} u_{21} + \ldots + \beta_{2n} u_{2n}.$$

Thus we rewrite v as

$$v = (\alpha_1 + \beta_1) u_1 + \ldots + (\alpha_k + \beta_k) u_k + \alpha_{11} u_{11} + \ldots + \alpha_{1m} u_{1m} + \beta_{21} u_{21} + \ldots + \beta_{2n} u_{2n},$$

and see that v is a linear combination of vectors in S.

Next we prove that S is an independent list: suppose that

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + \alpha_{11} u_{11} + \ldots + \alpha_{1m} u_{1m} + \beta_{21} u_{21} + \ldots + \beta_{2n} u_{2n} = \mathbf{0},$$

which we rewrite as

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_{21} u_{21} + \ldots + \beta_{2n} u_{2n} = -\alpha_{11} u_{12} - \ldots - \alpha_{1m} u_{1m}.$$

Now on the left-hand side of the equation above, we have a vector from U_2; on the right-hand side, we have a vector from U_1. Thus we see that

$$-\alpha_{11} u_{12} - \ldots - \alpha_{1m} u_{1m} \in U_1 \cap U_2,$$

so that we may write this vector as a linear combination of basis vectors for $U_1 \cap U_2$, say

$$-\alpha_{11} u_{12} - \ldots - \alpha_{1m} u_{1m} = \gamma_1 u_1 + \ldots + \gamma_k u_k.$$

However, the list

$$(u_1,\ u_2,\ \ldots,\ u_k,\ u_{11},\ \ldots,\ u_{1m})$$

is independent, so that

$$\gamma_1 = \ldots = \gamma_k = \alpha_{11} = \ldots = \alpha_{1m} = 0.$$

Returning to the vector

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_{21} u_{21} + \ldots + \beta_{2n} u_{2n} = -\alpha_{11} u_{12} - \ldots - \alpha_{1m} u_{1m} = \mathbf{0},$$

we see using the same reasoning that

$$\alpha_1 = \ldots = \alpha_k = \beta_{21} = \ldots = \beta_{2n} = 0.$$

Thus the list S is independent, as well as spanning, and is a basis for $U_1 + U_2$.

Recall that

$$S = S_1 \cup M = (u_1,\ u_2,\ \ldots,\ u_k,\ u_{11},\ \ldots,\ u_{1m},\ u_{21},\ \ldots,\ u_{2n}),$$

so that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Corollary. If U_1 and U_2 are subspaces of a finite dimensional vector space so that the sum $U_1 + U_2$ is direct, then

$$\dim(U_1 + U_2) = \dim(U_1 \oplus U_2) = \dim(U_1) + \dim(U_2).$$

Proof. Trivial by the previous theorem, since the intersection of summands in a direct sum is $\{\mathbf{0}\}$.