The Dimension of a Vector Space

We have spent a great deal of time and effort on understanding the geometry of vector spaces, but we have not yet discussed an important geometric idea—that of the size of the space.

For example, think about the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Which one is “bigger”? We have not defined precisely what we mean by “bigger” or “smaller”, but intuitively, you know that $\mathbb{R}^3$ is the larger space.

Now, the set $\mathcal{M}_2(\mathbb{R})$ of all $2 \times 2$ real matrices is also a vector space, so we could play the same game with it: which is the bigger vector space, $\mathcal{M}_2(\mathbb{R})$ or $\mathbb{R}^2$? This time, intuition fails us, because we are not used to thinking of $\mathcal{M}_2(\mathbb{R})$ spatially.

In this section, we will define precisely what we mean by the “size” of a vector space, thereby giving us the tools to answer such questions. In addition, we will see how the size of a vector space is closely related to linear independence and spanning.

Basis Size and Dimension

In the previous section, we saw that the set

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$. We also saw that the set

$$B_2 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^2$ as well.

It is interesting to note that both of the bases that we have found for $\mathbb{R}^2$ have two elements. Of course, there are many other bases for $\mathbb{R}^2$; is it possible to find a basis containing, say 3 vectors? Or a basis with only 1 vector? The following theorem answers this question in a surprising and mathematically beautiful way:

**Theorem 2.35.** If the finite dimensional vector space $V$ has a basis consisting of $n$ elements, then every basis for $V$ has $n$ elements.

Theorem 2.35 answers the question above in the negative: since we know of a basis for $\mathbb{R}^2$ that has two elements, then any basis that we can find for $\mathbb{R}^2$ has to have 2 elements. Similarly, we know that the set $\mathcal{M}_2(\mathbb{R})$ of all real $2 \times 2$ matrices has a basis consisting of the matrices

$$e_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad e_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

the theorem says that any other basis you can find for $\mathcal{M}_2(\mathbb{R})$ will also have 4 elements.
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**Proof of Theorem 2.35.** Let $S$ and $S'$ be bases for vector space $V$. Theorem 2.23 states that the length of an independent list is no more than the length of a spanning list. $S$ is independent and $S'$ is spanning, so

$$|S| \leq |S'|.$$  

Reversing the roles of $S$, and $S'$, we see that

$$|S'| \leq |S|.$$  

Thus

$$|S| = |S'|,$$

so that every basis for $V$ has the same number of elements.

The theorem now gives us an unambiguous way to define what we mean when we refer to the size of a vector space:

**Definition 2.36.** The *dimension* of a finite dimensional vector space $V$, denoted $\dim(V)$, is the number of vectors in a basis for $V$. We define the dimension of the trivial vector space (i.e., the vector space containing only $0$) to be 0.

In a sense, the dimension of a vector space tells us how many vectors are needed to “build” the space, thus gives us a way to compare the relative sizes of the spaces. Using the theorem, it is clear that

- $\dim(\mathbb{R}^n) = n$
- $\dim(s\mathfrak{l}(2, \mathbb{R})) = 3$
- $\dim(M_{mn}(\mathbb{F})) = mn$
- $\dim(P_n(\mathbb{F})) = n + 1$
- $\dim(U_2(\mathbb{R})) = 3$.

In particular, our original observation that $\mathbb{R}^3$ is a “larger” space than is $\mathbb{R}^2$ is correct (and now defined more precisely). Similarly, we can conclude that $\mathcal{M}_2(\mathbb{R})$ is a larger space that $\mathbb{R}^3$, as it takes more vectors to build the space.

With the previous theorem in mind, we take a moment to consider the dimensions of subspaces. Recall that a *subspace* of a vector space $V$ is a subset $U$ of the vectors of $V$, which is a vector space in its own right (with the same operations as $V$).

For example, the vector space $\mathcal{U}_2(\mathbb{R})$ is a subspace of $\mathcal{M}_2(\mathbb{R})$. Of course, we expect $\mathcal{U}_2(\mathbb{R})$ to be relatively small compared to the ambient space, thus to have smaller dimension (as confirmed by the list above). The following theorem confirms this observation in general:

**Theorem 2.38.** If $U$ is a subspace of a finite dimensional vector space $V$, then

$$\dim(U) \leq \dim(V).$$
Proof. A basis $S$ for $U$ consists of linearly independent vectors in $V$, and a basis $S'$ for $V$ is a spanning list. Recalling Theorem 2.23, which states that the length of an independent list in $V$ is never more than the length of a spanning list in $V$, we see that

$$|S| \leq |S'|.$$  

Understanding Bases

At this point, it should be clear that a basis for a vector space $V$ has to have enough vectors to be able to span $V$, but not so many that it is no longer linearly independent. We make this idea precise with the following theorem:

**Theorem.** Let $V$ be an $n$-dimensional vector space, that is, every basis of $V$ consists of $n$ vectors. Then

(a) Any list of vectors from $V$ containing more than $n$ vectors is linearly dependent.

(b) Any list of vectors from $V$ containing fewer than $n$ vectors does not span $V$.

The theorem is really just a restatement (in terms of bases) of a result that we already know: for part (a), we recall that the length of an independent list is no longer than the length of a spanning list, so clearly a list that is longer than the length of a list of basis vectors must be dependent. Similarly for part (b), a list that is shorter than the length of a basis cannot be spanning.

**Key Point.** Adding too many vectors to a set will force the set to be linearly dependent; on the other hand, taking too many vectors away from a set will prevent it from spanning. A basis is a sort of “sweet spot” between linear independence and spanning: if a basis $S$ for $V$ has $n$ elements and we remove one, then $S$ no longer spans; add one, and $S$ is no longer linearly independent.

As an example, we know that the list

$$B = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis for $\mathbb{R}^3$; it is linearly independent, and it spans $\mathbb{R}^3$. If we remove the last vector, to get the set

$$B' = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

then we no longer have a spanning list for $\mathbb{R}^3$. Indeed, this is easy to see geometrically: the two vectors from $B'$ are graphed below in $\mathbb{R}^3$, and clearly cannot be used to describe the vertical component of the space:
In fact, these two vectors span the \textit{subspace} of $\mathbb{R}^3$ shaded in gray, but not all of $\mathbb{R}^3$.

The theorems above lead to some important facts about the geometry of vector spaces and their subspaces.

\textbf{Theorems 2.39/2.42.} Let $V$ be an $n$-dimensional vector space.

1. Any independent list of $n$ vectors in $V$ is a basis.
2. Any spanning list of $n$ vectors in $V$ is a basis.

\textbf{Key Point.} We know that a basis for a vector space must be a linearly independent spanning set; to conclude that a set is a basis, we must be certain that both conditions are met.

However, Theorems 2.39 and 2.42 make it much easier to determine whether or not a set is a basis: if a set has the right number of vectors—the same as the dimension of $V$—then we can quickly check to see if the set is a basis by determining if it is a linearly independent set, or alternatively by checking that the set spans $V$. We don’t have to check \textit{both} conditions anymore, just one of them!

\textbf{Proof of Theorems 2.39/2.42.} 1. Suppose that $V$ is an $n$-dimensional vector space, and let $S$ be a list of $n$ linearly independent vectors in $V$. If $S$ is not a basis, then it may be extended to a basis $S'$ by Theorem 2.33. However, every basis of $V$ has $n$ vectors, so the extension to $S'$ was trivial, so that $S$ must be a basis for $V$.

2. Let $S$ be a spanning list consisting of $n$ vectors. If $S$ is not a basis, then it must contain a basis by Theorem 2.31. However, any sublist of $S$ obtained nontrivially has fewer than $n$ vectors and cannot be a basis. Thus $S$ must already be a basis for $V$. 
Let’s use the theorems to investigate the example introduced above: the set

\[ B' = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right), \]

while linearly independent in \( \mathbb{R}^3 \), does not span \( \mathbb{R}^3 \):

However, if we extend \( B' \) by adding any linearly independent vector, we will have a list of the right length, and thus a basis. In other words, we can add any vector we like to \( B' \) (as long as that vector is not already in the span of \( B' \)), and we will then have a basis. In this case, we just need to choose a vector that does not lie in the \( xy \) plane—for example, we could choose

\[ (2, 1, 1) \]
It is easy to check that the list
\[
\left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)
\]
is linearly independent, and thus a basis for \( \mathbb{R}^3 \), as \( \mathbb{R}^3 \) is 3 dimensional. Since the set is independent and has the right number of vectors, Theorem 2.39 tells us that we don’t have to check that it spans \( \mathbb{R}^3 \) to know that it’s a basis!

Alternatively, we could choose the vector
\[
(0, 3, -1)
\]
to add to \( B' \):

Again, it is easy to check that the list
\[
\left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right)
\]
is linearly independent, so the theorem indicates that it is also a basis.

**Example**

The vectors 
\[
f_1(x) = 2x - 3, \ f_2(x) = x^2 + 1, \ \text{and} \ f_3(x) = 2x^2 - x
\]
are linearly independent. Complete the list to form a basis for $P_3(\mathbb{R})$, the space of all real-valued polynomials of degree no more than 3.

We know that a basis for $P_3(\mathbb{R})$ must consist of 4 linearly independent vectors that span $P_3(\mathbb{R})$. Since we already have 3 linearly independent vectors, we simply need to find one more independent vector to add to the list.

It is not too difficult to find a vector independent from those in our current list—since none of the polynomials in our list include a term of $x^3$, $f_4(x) = x^3$ seems like a good choice. Indeed, it is clear that $x^3$ is not a linear combination of $f_1$, $f_2$, and $f_3$, so the list

$$(2x - 3, x^2 + 1, 2x^2 - x, x^3)$$

of four independent vectors in the 4-dimensional vector space $P_3(\mathbb{R})$ is a basis.

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**Dimensions of Sums and Direct Sums**

Finally, we record a theorem that will allow us to quickly calculate the dimension of a sum of subspaces.

**Theorem 2.43.** If $U_1$ and $U_2$ are subspaces of a vector space $V$, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

**Remark.** Recall from a previous homework assignment that the intersection of a pair of subspaces is also a subspace (and thus a vector space). Thus the notation $\dim(U_1 \cap U_2)$ is justified.

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**Proof.** Let $S' = (u_1, u_2, \ldots, u_k)$ be a basis for $U_1 \cap U_2$. Of course, $S'$ is independent, so by Theorem 2.33, we may extend $S'$ to bases

$$S_1 = (u_1, u_2, \ldots, u_k, u_{11}, \ldots, u_{1m})$$

and

$$S_2 = (u_1, u_2, \ldots, u_k, u_{21}, \ldots, u_{2n})$$

for $U_1$ and $U_2$, respectively. Set

$$M = S_2 - S_1 \cap S_2 = (u_{21}, \ldots, u_{2n}).$$

Now I claim that the list

$$S = S_1 \cup M = (u_1, u_2, \ldots, u_k, u_{11}, \ldots, u_{1m}, u_{21}, \ldots, u_{2n})$$

is a basis for $U_1 + U_2$. First, let us prove that $S$ spans $U_1 + U_2$: if $v \in U_1 + U_2$, then $v$ may be written as

$$v = u'_1 + u'_2,$$
Thus the list \( S \) so that we see using the same reasoning that

\[
\text{Returning to the vector } U \text{ so that we may write this vector as a linear combination of basis vectors for } U \text{ side, we have a vector from } U \text{; on the right-hand side, we have a vector from } U_1. \text{ Thus we see that}
\]

\[
-\alpha_{11}u_{12} - \ldots - \alpha_{1m}u_{1m} \in U_1 \cap U_2,
\]

so that we may write this vector as a linear combination of basis vectors for \( U_1 \cap U_2 \), say

\[
-\alpha_{11}u_{12} - \ldots - \alpha_{1m}u_{1m} = \gamma_1u_1 + \ldots + \gamma_ku_k.
\]

However, the list

\[
(u_1, u_2, \ldots, u_k, u_{11}, \ldots, u_{1m})
\]

is independent, so that

\[
\gamma_1 = \ldots = \gamma_k = \alpha_{11} = \ldots = \alpha_{1m} = 0.
\]

Returning to the vector

\[
\alpha_1u_1 + \ldots + \alpha_ku_k + \beta_{21}u_{21} + \ldots + \beta_{2n}u_{2n} = -\alpha_{11}u_{12} - \ldots - \alpha_{1m}u_{1m} = 0,
\]

we see using the same reasoning that

\[
\alpha_1 = \ldots = \alpha_k = \beta_{21} = \ldots = \beta_{2n} = 0.
\]

Thus the list \( S \) is independent, as well as spanning, and is a basis for \( U_1 + U_2 \).

Recall that

\[
S = S_1 \cup M = (u_1, u_2, \ldots, u_k, u_{11}, \ldots, u_{1m}, u_{21}, \ldots, u_{2n}),
\]

so that

\[
\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).
\]

**Corollary.** If \( U_1 \) and \( U_2 \) are subspaces of a finite dimensional vector space so that the sum \( U_1 + U_2 \) is direct, then

\[
\dim(U_1 + U_2) = \dim(U_1 \oplus U_2) = \dim(U_1) + \dim(U_2).
\]

**Proof.** Trivial by the previous theorem, since the intersection of summands in a direct sum is \( \{0\} \).