Series Summary

Well-known convergent series:

1. Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^n + ... = \frac{a}{1-r}$ for $|r| < 1$.

2. P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...$ for $p > 1$.

Well-known divergent series:

1. Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^n + ...$ for $|r| \geq 1$.

2. P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...$ for $p \leq 1$.

General strategy for choosing a test for convergence:

1. If the series has terms of the form ar^{n-1}, the series is geometric and the convergence of the series depends on the value for r.

2. If the series has terms of the form $\frac{1}{n^p}$ where p is a constant, the series is a p-series, and the convergence of the series depends on the value for p.

3. If the terms of the series are positive and $a_n = f(n)$ for some continuous function $f(x)$, *and* the improper integral of $f(x)$ can be evaluated, use the integral test.

4. If the limit $\lim_{n\to\infty} a_n$ can be easily determined, try the limit test for divergence.

5. If the terms of the series are positive and the series can be easily compared to a geometric series or to a p-series by simplifying the terms, try one (or both) of the comparison tests.

6. If the series involves factorials (or possibly other products or constant bases raised to the nth power), try the ratio test.

7. If the series does not have factorials but contains terms raised to the nth power, try the root test.

8. If the series has terms involving $(-1)^n$, try the alternating series test.

Convergence and Divergence Tests

1. n^{th} **Term Test for Divergence**: $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.

2. **Integral Test**: Suppose that $\{a_n\}$ is a sequence of positive terms. If $a_n = f(n)$, where f is a continuous, positive, decreasing function then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)dx$ both converge or both diverge.

3. **Comparison Test**: Let $\sum a_n$ be a series with nonnegative terms.
   (a) $\sum a_n$ converges if there is a convergent series $\sum b_n$ with $a_n \leq b_n$ for all n.
   (b) $\sum a_n$ diverges if there is a divergent series $\sum b_n$ with $a_n \geq b_n$ for all n.

4. **Limit Comparison Test**: If $a_n > 0$ and $b_n > 0$ for all $n \geq N$, N an integer, and if
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$
   where $c > 0$ is a finite number, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

5. **Alternating Series Test**: The alternating series
$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$
   converges if all three of the following conditions are satisfied:
   (a) Each a_i is positive
   (b) $a_n \geq a_{n+1}$ for all $n \geq N$, for some integer N
   (c) $\lim_{n\to\infty} a_n = 0$.

6. **Absolute Convergence Test**: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ does as well.

7. **Ratio Test**: Let $\sum a_n$ be a series so that
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$
   (a) If $\rho > 1$ or $\rho = \infty$, the series $\sum a_n$ *diverges.*
   (b) If $\rho < 1$, the series $\sum a_n$ *converges absolutely* (i.e., $\sum a_n$ and $\sum |a_n|$ converge).
   (c) If $\rho = 1$, the test is inconclusive.

8. **Root Test**: Let $\sum a_n$ be a series, and suppose that
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$
   (a) If $\rho > 1$ or if $\rho = \infty$, the series $\sum a_n$ *diverges.*
   (b) If $\rho < 1$, the series *converges absolutely.*
   (c) If $\rho = 1$, the test is inconclusive.

Test	Applies	Can Tell Us
N-th term test	any series	only divergence
Integral test	terms are nonnegative and can be integrated	convergence or divergence
Direct comparison test	nonnegative terms	convergence or divergence (sometimes)
Limit comparison test	nonnegative terms	convergence or divergence (sometimes)
Alternating series test	only alternating series	only convergence
Absolute convergence test	any series	only absolute convergence
Root test/ Ratio test	any series	absolute convergence or divervgence (sometimes)