Series Summary

Well-known convergent series:

1. Geometric Series: \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^n + \ldots = \frac{a}{1-r} \) for \(|r| < 1\).

2. P-series: \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots \) for \( p > 1 \).

Well-known divergent series:

1. Geometric Series: \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^n + \ldots \) for \(|r| \geq 1\).

2. P-series: \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots \) for \( p \leq 1 \).

General strategy for choosing a test for convergence:

1. If the series has terms of the form \( ar^{n-1} \), the series is geometric and the convergence of the series depends on the value for \( r \).

2. If the series has terms of the form \( \frac{1}{n^p} \) where \( p \) is a constant, the series is a p-series, and the convergence of the series depends on the value for \( p \).

3. If the terms of the series are positive and \( a_n = f(n) \) for some continuous function \( f(x) \), and the improper integral of \( f(x) \) can be evaluated, use the integral test.

4. If the limit \( \lim_{n \to \infty} a_n \) can be easily determined, try the limit test for divergence.

5. If the terms of the series are positive and the series can be easily compared to a geometric series or to a p-series by simplifying the terms, try one (or both) of the comparison tests.

6. If the series involves factorials (or possibly other products or constant bases raised to the \( n \)th power), try the ratio test.

7. If the series does not have factorials but contains terms raised to the \( n \)th power, try the root test.

8. If the series has terms involving \( (-1)^n \), try the alternating series test.
### Series Summary

#### Convergence and Divergence Tests

1. **$n^{th}$ Term Test for Divergence**: $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} a_n \neq 0$.

2. **Integral Test**: Suppose that $\{a_n\}$ is a sequence of positive terms. If $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)dx$ both converge or both diverge.

3. **Comparison Test**: Let $\sum a_n$ be a series with nonnegative terms.
   - (a) $\sum a_n$ converges if there is a convergent series $\sum b_n$ with $a_n \leq b_n$ for all $n$.
   - (b) $\sum a_n$ diverges if there is a divergent series $\sum b_n$ with $a_n \geq b_n$ for all $n$.

4. **Limit Comparison Test**: If $a_n > 0$ and $b_n > 0$ for all $n \geq N$, $N$ an integer, and if $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where $c > 0$ is a finite number, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

5. **Alternating Series Test**: The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges if all three of the following conditions are satisfied:
   - (a) Each $a_i$ is positive
   - (b) $a_n \geq a_{n+1}$ for all $n \geq N$, for some integer $N$
   - (c) $\lim_{n \to \infty} a_n = 0$.

6. **Absolute Convergence Test**: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ does as well.

7. **Ratio Test**: Let $\sum a_n$ be a series so that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$.
   - (a) If $\rho > 1$ or $\rho = \infty$, the series $\sum a_n$ diverges.
   - (b) If $\rho < 1$, the series $\sum a_n$ converges absolutely (i.e., $\sum a_n$ and $\sum |a_n|$ converge).
   - (c) If $\rho = 1$, the test is inconclusive.

8. **Root Test**: Let $\sum a_n$ be a series, and suppose that $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \rho$.
   - (a) If $\rho > 1$ or if $\rho = \infty$, the series $\sum a_n$ diverges.
   - (b) If $\rho < 1$, the series converges absolutely.
   - (c) If $\rho = 1$, the test is inconclusive.
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