Section 11.6

Absolute and Conditional Convergence, Root and Ratio Tests

In this chapter we have seen several examples of convergence tests that only apply to series whose terms are nonnegative. In this section, we will learn a test that will in some cases allow us to use the same tools for series with positive and negative terms. We begin by recording a definition:

**Definition.** A series $\sum a_n$ **converges absolutely** if the series of absolute values $\sum |a_n|$ converges. A series **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ does not converge.

We have already seen that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges, but the harmonic series

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

does not. Since the alternating harmonic series converges, but the series we get when we change all of its terms to positives does not, we say that the alternating harmonic series converges conditionally.

On the other hand, the series

$$\sum_{n=1}^{\infty} (-1)^n \left( \frac{2}{3} \right)^n$$

converges absolutely since the series of absolute values

$$\sum_{n=1}^{\infty} \left| (-1)^n \left( \frac{2}{3} \right)^n \right| = \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n$$

is geometric with $r = \frac{2}{3} < 1$.

It turns out that any series that is absolutely convergent must be convergent itself:

**Absolute Convergence Test.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ does as well.

In other words, if the new series we get from $\sum a_n$ by making all of its terms positive is a convergent series, then the original series converges as well. This seems plausible—it is more likely that a series with both positive and negative terms will converge (since many of the terms will effectively cancel), so if the series with all positive terms converges, so must the original series.
Section 11.6

There are several important points to note here: first of all, the theorem is helpful since we already have several tests for series that have positive terms. To test a series that doesn’t have all positive terms, we can take absolute values and use one of the previously studied convergence tests that apply to series with positive terms; if the series of absolute values converges, then the original series does too.

If the series is not absolutely convergent, it may still be conditionally convergent, or it may diverge after all. To check, try the Alternating Series Test (if the original series was alternating) or the nth term test.

Example. Determine if each of the following series converges absolutely, conditionally, or diverges.

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\tan(\frac{1}{n})}{n^2}
\]

To determine if the series converges absolutely, we need to determine if

\[
\sum_{n=1}^{\infty} \left| (-1)^n \frac{\tan(\frac{1}{n})}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\tan(\frac{1}{n})}{n^2}
\]

converges or diverges.

Let’s try the limit comparison test. For large \( n \), \( \frac{1}{n} \to 0 \) so that \( \tan\left(\frac{1}{n}\right) \to 0 \). It seems that the function \( \tan\left(\frac{1}{n}\right) \) behaves similarly to \( \frac{1}{n} \), so we will replace the numerator with \( \frac{1}{n} \). We compare

\[
\frac{\tan\left(\frac{1}{n}\right)}{n^2}
\]

to

\[
\frac{1}{n^2} = \frac{1}{n^3}
\]

: \( \lim_{n \to \infty} \frac{\tan\left(\frac{1}{n}\right)}{n^3} = \lim_{n \to \infty} \frac{n^3 \tan\left(\frac{1}{n}\right)}{n^2} = \lim_{n \to \infty} n \tan\left(\frac{1}{n}\right) \).

Notice that the last limit above has indeterminate form \( 0 \cdot \infty \); thus we rewrite it so that L'Hopital’s Rule applies:

\[
\lim_{n \to \infty} n \tan\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{n \tan\left(\frac{1}{n}\right)}{1} = \lim_{n \to \infty} \frac{-\frac{\sec^2\left(\frac{1}{n}\right)}{n^2}}{-\frac{1}{n^2}} \]

\[
= \lim_{n \to \infty} \frac{\sec^2\left(\frac{1}{n}\right)}{1} = \sec^2 0 = 1.
\]
The comparison series $\frac{1}{n^3}$ converges since it is a $p$-series with $p = 3$; since the limit of the quotient of the terms is 1,

$$\sum_{n=1}^{\infty} \frac{\tan\left(\frac{1}{n}\right)}{n^2}$$

converges as well by the limit comparison test.

Since the series of absolute values converges (i.e. is the original series converges absolutely), we conclude that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\tan\left(\frac{1}{n}\right)}{n^2}$$

converges as well, this time by the absolute comparison test.

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Example. Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n} - 1}$$

converges absolutely, converges conditionally, or diverges.

It is easy to see that the series does not converge absolutely by the $n$th term test:

$$\lim_{n \to \infty} \left| (-1)^n \frac{\sqrt{n}}{\sqrt{n} - 1} \right| = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} - 1}$$

$$= \lim_{n \to \infty} \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}} - 1}$$

$$= \lim_{n \to \infty} \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}} - 1}$$

$$= \lim_{n \to \infty} 1$$

$$= 1.$$

Since the limit of the terms is non-zero, the series of absolute values diverges.

The original series may still converge (i.e., the series might converge conditionally), or it may diverge; however, the result of the previous test indicate that we should try the $n$th term test on the original terms. If $n$ is even, we already know that

$$\lim_{n \to \infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n} - 1} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} - 1}$$

$$= 1.$$
Section 11.6

If \( n \) is odd, then

\[
\lim_{n \to \infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n} - 1} = \lim_{n \to \infty} -\frac{\sqrt{n}}{\sqrt{n} - 1} = -1.
\]

Thus the general limit does not exist, so that the series fails the \( n \)th term test, and diverges.

Example. Determine whether the series

\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n^n}
\]

converges absolutely, converges conditionally, or diverges.

Let’s start by testing for absolute convergence; we need to look at the series

\[
\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{\ln n^n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}.
\]

We know that we can integrate \( \frac{1}{n \ln n} \) by u-substitution, so we try the integral test. Setting

\[
u = \ln n, \text{ so that } \frac{1}{n} \, dn = du,
\]

we have

\[
\int \frac{1}{n \ln n} \, dn = \int \frac{1}{u} \, du = \ln u + C = \ln(\ln n) + C.
\]

So

\[
\int_{2}^{\infty} \frac{1}{n \ln n} \, dn = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{n \ln n} \, dn = \lim_{b \to \infty} \left( \ln(\ln n) \right|_{2}^{b} = \lim_{b \to \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty.
\]
Section 11.6

Since the improper integral diverges, the series of absolute values does too. Thus the original series is not absolutely convergent, but might still be conditionally convergent. To check, let’s try using the Alternating Series Test. We have

\[ a_n = \frac{1}{n \ln n}; \]

each term \( a_n \) is nonnegative, and it is clear that

\[ \frac{1}{(n + 1) \ln(n + 1)} < \frac{1}{n \ln n} \]

since \((n + 1) \ln(n + 1) > n \ln n\).

Finally, we note that

\[ \lim_{n \to \infty} \frac{1}{n \ln n} = 0. \]

Since the limit is 0, the original series

\[ \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \]

converges by the alternating series test. Since it does not converge absolutely, we specify that the series converges conditionally.

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Root and Ratio Tests

The Ratio Test is a convergence test that can be particularly helpful when the series at hand has terms involving factorials or \( n \)th powers:

**Ratio Test.** Let \( \sum a_n \) be a series so that

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho. \]

1. If \( \rho > 1 \) or \( \rho = \infty \), the series \( \sum a_n \) diverges.
2. If \( \rho < 1 \), the series \( \sum a_n \) converges absolutely (i.e., \( \sum a_n \) and \( \sum |a_n| \) converge).
3. If \( \rho = 1 \), the test is inconclusive.

The Root Test is generally helpful if terms of the series involve \( n \)th powers; it is *not* usually helpful when there are factorials involved.

**Root Test.** Let \( \sum a_n \) be a series, and suppose that

\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = \rho. \]
Section 11.6

1. If $\rho > 1$ or if $\rho = \infty$, the series $\sum a_n$ diverges.

2. If $\rho < 1$, the series converges absolutely.

3. If $\rho = 1$, the test is inconclusive.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} n \left(\frac{3}{2}\right)^n$$

converges or diverges.

The $n$th power in the fraction above indicates that the root test could be a good way to proceed. As we evaluate the limit, recall that

$$\lim_{n \to \infty} n^{1/n} = 1.$$

$$\lim_{n \to \infty} \sqrt[n]{n \left(\frac{3}{2}\right)^n} = \lim_{n \to \infty} n^{1/n} \left(\frac{3}{2}\right) = \frac{3}{2}$$

Since the limit is greater than 1, the series diverges by the root test.

Notice that the integral test would not be helpful in this example, since we do not know how to integrate a function of the form $f(x) = x^{(\frac{3}{2})^x}$.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

converges or diverges.

The terms of the series have a factorial in them, so the ratio test seems like a good place to start. Since all of the terms are positive, we can ignore the absolute value signs, and evaluate the limit.
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{(n+1)!}{n!}}{\frac{e^{(n+1)^2}}{e^{n^2}}} = \lim_{n \to \infty} \frac{(n+1)!e^{n^2}}{n!e^{(n+1)^2}}.
\]

Now the fraction above may be rewritten; since
\[
\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1}{n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1} = n + 1,
\]
we have
\[
\lim_{n \to \infty} \frac{(n+1)!e^{n^2}}{n!e^{(n+1)^2}} = \lim_{n \to \infty} \frac{(n+1)e^{n^2}}{e^{n^2+2n+1}} = \lim_{n \to \infty} \frac{(n+1)}{e^{2n+1}} = LR \lim_{n \to \infty} \frac{1}{2e^{2n+1}} = 0
\]
since
\[
\lim_{n \to \infty} 2e^{2n+1} = \infty.
\]
Since the limit is 0, the series converges by the ratio test.

**Example.** Determine if the series
\[
\sum_{n=1}^{\infty} \frac{n!}{n^n}
\]
converges or diverges.

It seems reasonable to guess that either the root test or the ratio test will be appropriate for this example. Notice that if we try the root test, we will need to handle terms of the form \(\sqrt[n]{n!}\), which will not readily simplify. Thus our best choice here is probably the ratio test. We need to evaluate
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{n!(n+1)^{n+1}}
= \lim_{n \to \infty} \frac{n^n(n+1)!}{n!(n+1)^{n+1}}
= \lim_{n \to \infty} \frac{n^n(n+1)}{(n+1)^{n+1}}
= \lim_{n \to \infty} \frac{n^n}{(n+1)^n}
= \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n.
\]

The limit looks unfathomable, but a simple trick will help us evaluate it:

\[
\lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \to \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}
= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}
= \frac{1}{e}
\]

since

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,
\]

as we saw in section 11.1.

Since \(e > 1\), we know that \(\frac{1}{e} < 1\); so the series converges by the ratio test.