Alternating Series Test

As we have seen several times in this chapter, certain types of series have special forms that make them easier to work with; *alternating series* are one such type of series. A series is alternating if the sign of the terms alternate between positive and negative, such as in the *alternating harmonic series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This series differs from the usual harmonic series,

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots ,$$

since the signs in the former series alternate.

Alternating series have a special form; since the sign alternates from term to term, any alternating series can be written as

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n,$$

where $a_n > 0$; the factor of $(-1)^{n-1}$ takes care of the alternating sign.

The following simple test can often be used to determine that an alternating series converges:

Alternating Series Test. The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. Each b_i is positive
2. $b_n \geq b_{n+1}$ for all n
3. $\lim_{n\to\infty} b_n = 0$.

Be careful to note that the test can *only* confirm that a series converges, but cannot be used to tell us that a series diverges.

In a sense, this test is *similar* to the nth term test, as it involves taking the limit of the terms. However, the Alternating Series Test *only* applies to alternating series, while the nth term test can be applied to any series, including alternating series. It is extremely important to use the test you have chosen correctly–for example, if you wish to test the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

for convergence via the alternating series test, you should evaluate

$$\lim_{n\to\infty} a_n;$$

on the other hand, if you wish to test the same series for *divergence* using the nth term test, you should evaluate

$$\lim_{n\to\infty} (-1)^n a_n.$$

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ fails condition 3 of the Alternating Series Test, i.e.

$$\lim_{n\to\infty} a_n \neq 0,$$

then

$$\lim_{n\to\infty} (-1)^{n-1} a_n \neq 0,$$

which means that the series itself will fail the nth term test and must diverge.

Example. Show that the alternating harmonic series converges.

The alternating harmonic series, given by

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

has $a_n = \frac{1}{n}$. Each $a_i > 0$, and it is easy to see that $a_n \geq a_{n+1}$ for all n since $\frac{1}{n} > \frac{1}{n+1}$. Finally, since

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

we see that the alternating harmonic series satisfies the three conditions of the theorem, thus converges by the alternating series test.

Example. Determine if the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+3}$$

converges or diverges.

Since the series is alternating, the alternating series test seems like a natural choice for testing it. We evaluate the limit of the *positive* part of the summand:

$$\begin{aligned} \lim_{n\to\infty} \frac{n^2}{n^2+5} &= \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^2}} \\ &= 1. \end{aligned}$$

Since the limit is *not* 0, the alternating series test cannot provide us with a conclusion.

However, the result above is indicative of what will happen when we try the nth term test for divergence. Indeed, it is clear that

$$\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+5}$$

does not exist, since the values oscillate between 1 and -1 as n goes towards infinity. The series fails the nth term test, and diverges.