Section 11.4

Comparison Tests

In Chapter 7, we learned that we can sometimes determine whether or not an improper integral converges by comparing it to an improper integral whose convergence or divergence has already been established. In this section, we will learn similar comparison tests for determining the convergence or divergence of series.

The Direct Comparison Test for series uses the same concept as does the direct comparison test for improper integrals:

**Direct Comparison Test.** Let \( \sum a_n \) be a series with nonnegative terms.

1. \( \sum a_n \) converges if there is a convergent series \( \sum b_n \) with \( a_n \leq b_n \) for all \( n \).
2. \( \sum a_n \) diverges if there is a divergent series \( \sum b_n \) with \( a_n \geq b_n \) for all \( n \).

In other words, if a series with relatively large terms (compared with terms \( a_n \)) converges, then \( \sum a_n \) must converge as well. On the other hand, if a series with relatively small terms in comparison with the terms \( a_n \) diverges, \( \sum a_n \) must diverge.

Choosing a comparison series can be difficult; one thing to keep in mind is that simplifying the form of \( a_n \) may be helpful. We generally accomplish this simplification by comparing to a geometric series or to a \( p \)-series.

A useful fact to keep in mind when working with series whose terms involve the natural log function is that

\[
\ln x < x
\]

for all \( x \); this fact can often provide us with a natural comparison function.

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**Example.** Use the direct comparison test to determine if

\[
\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}
\]

converges or diverges.

The series looks very similar to the geometric series

\[
\sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n,
\]

so a direct comparison seems like a good idea. Since

\[
\sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n
\]

is a geometric series with \( r = \frac{3}{2} > 1 \), we know that it diverges; we also know that

\[
\frac{4 + 3^n}{2^n} > \frac{3^n}{2^n}
\]
so

\[\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}\]

diverges by the direct comparison test.

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**Limit Comparison Test**

We were able to determine that the series

\[\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}\]

diverges because of the easy comparison with the diverging series

\[\sum_{n=1}^{\infty} \frac{3^n}{2^n};\]

if, however, we wished to compare the series

\[\sum_{n=1}^{\infty} \frac{-4 + 3^n}{2^n},\]

we would be at a loss: since

\[-\frac{4 + 3^n}{2^n} < \frac{3^n}{2^n},\]

and the series

\[\sum_{n=1}^{\infty} \frac{3^n}{2^n}\]

diverges, the direct comparison test fails to provide a conclusion.

Fortunately, there is another comparison test for just such occasions:

**Limit Comparison Test.** Suppose that \(a_n > 0\) and \(b_n > 0\) for all \(n\). If

\[\lim_{n \to \infty} \frac{a_n}{b_n} = c,\]

where \(c > 0\) is a finite number, then \(\sum a_n\) and \(\sum b_n\) both converge or both diverge.

It is generally helpful to try comparing the series \(\sum a_n\) to a series that is *similar to but simpler than* \(\sum a_n\). Keep in mind that using the theorem will require us to evaluate a limit—thus it is important to end up with a limit that we can actually evaluate. We will often use L'Hopital’s rule to evaluate the limits that show up in these problems.

It can also be helpful to try to determine how the \(a_n\) behave for large \(n\). For instance, \(n^2 + 1\) behaves almost identically to \(n^2\), and the replacement might be useful.

Finally, if comparing \(\sum a_n\) to \(\sum b_n\) using one of the comparison tests doesn’t provide any useful information, it may be helpful to compare the same two series using the other comparison test.
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Example. Use the limit comparison test to show that
\[
\sum_{n=1}^{\infty} \frac{-4 + 3^n}{2^n}
\]
diverges.

As we have already discussed, a direct comparison with the series whose terms are
\[a_n = \frac{3^n}{2^n}\]
is unhelpful here, since
\[
\frac{-4 + 3^n}{2^n} < \frac{3^n}{2^n}
\]
and
\[
\sum_{n=1}^{\infty} \frac{3^n}{2^n}
\]
diverges.

Let’s try using the limit comparison test with the same comparison function:
\[
\lim_{n \to \infty} \frac{-4 + 3^n}{2^n} = \lim_{n \to \infty} \frac{2^n(-4 + 3^n)}{3^n 2^n} = \lim_{n \to \infty} \frac{-4 + 3^n}{3^n} = LR = \lim_{n \to \infty} \frac{3^n \ln 3}{3^n \ln 3} = \lim_{n \to \infty} \frac{1}{1} = 1.
\]

Since the limit of the quotient is finite, the series behave the same way (as we might have already suspected):
\[
\sum_{n=1}^{\infty} \frac{3^n}{2^n}
\]
diverges, so
\[
\sum_{n=1}^{\infty} \frac{-4 + 3^n}{2^n}
\]
does as well, by the limit comparison test.

Example. Determine if
\[
\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}
\]
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converges or diverges.

We might wish to solve this problem using a simple comparison; we simplify \( \ln n \) by replacing it with \( n \). Thus our comparison function is

\[
\frac{1}{1 + n}.
\]

Using the limit comparison test, we evaluate

\[
\lim_{n \to \infty} \frac{\frac{1}{1 + \ln n}}{\frac{1}{1 + n}} = \lim_{n \to \infty} \frac{1 + n}{1 + \ln n} = \lim_{n \to \infty} \frac{1}{\frac{1}{n}} = \lim_{n \to \infty} n = \infty.
\]

Since the limit is infinite, the limit comparison test does not provide us with any information.

We might wish to try the comparison again, this time using the *direct comparison test* instead. Since \( \ln n < n \),

\[
\frac{1}{1 + \ln n} > \frac{1}{1 + n}.
\]

Now

\[
\sum_{n=1}^{\infty} \frac{1}{1 + n}
\]

is a diverging series (which we can see using a limit comparison test with the harmonic series); thus

\[
\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}
\]

diverges as well, by the direct comparison test.

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**Example.** Determine if

\[
\sum_{n=1}^{\infty} \frac{1}{n^{10} - 1}
\]

converges or diverges.

It seems natural to try a comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^{10}} \), which is a \( p \)-series with \( p = 10 \), thus is convergent. However, since

\[
n^{10} - 1 < n^{10},
\]

it is clear that

\[
\frac{1}{n^{10} - 1} < \frac{1}{n^{10}}.
\]
since the convergent series is smaller than the series in question, we gain no information from a direct comparison. But the two functions behave nearly identically, so we expect
\[ \sum_{n=1}^{\infty} \frac{1}{n^{10} - 1} \]
to converge. Let’s compare the same series using the limit comparison test. We need to evaluate
\[
\lim_{n \to \infty} \frac{\frac{1}{n^{10} - 1}}{\frac{1}{n^{10}}} = \lim_{n \to \infty} \frac{n^{10}}{n^{10} - 1} = 1.
\]
So since
\[ \sum_{n=1}^{\infty} \frac{1}{n^{10}} \]
converges,
\[ \sum_{n=1}^{\infty} \frac{1}{n^{10} - 1} \]
does as well, by the limit comparison test.

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**Example.** Determine if
\[ \sum_{n=1}^{\infty} \frac{n - 1}{n^2 \sqrt{n} + n} \]
converges or diverges.

For large \( n \), the numerator of the fraction behaves similarly to \( n \), while the denominator behaves like
\[ n^2 \sqrt{n} = n^{\frac{5}{2}}. \]
So let’s compare
\[ \frac{n - 1}{n^2 \sqrt{n} + n} \]
to
\[ \frac{n}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{3}{2}}}: \]
\[
\lim_{n \to \infty} \frac{n - 1}{n^{\frac{5}{2}} + n} = \lim_{n \to \infty} \frac{n^{\frac{3}{2}}(n - 1)}{n^2 \sqrt{n} + n} = \lim_{n \to \infty} \frac{n^{\frac{5}{2}} - n^{\frac{3}{2}}}{n^2 + n} = \lim_{n \to \infty} \frac{1 - \frac{1}{n^{\frac{2}{2}}}}{1 + \frac{1}{n^2}} = 1.
\]
Since
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]
is a \( p \)-series with \( p > 1 \), it converges; thus
\[ \sum_{n=1}^{\infty} \frac{n - 1}{n^2 \sqrt{n} + n} \]
converges as well, by the limit comparison test.