1. Consider the vectors x^2, $2x-1$, and x^2+1 in $\mathcal{P}_2(\mathbb{R})$.

   (a) Do the vectors span $\mathcal{P}_2(\mathbb{R})$? If so, show it; if not, provide an example of a vector not in their span.

   Solution: Every linear combination of the vectors above has form

$$\alpha x^2 + 2\beta x - \beta + \gamma x^2 + \gamma \quad = \quad (\alpha+\gamma)x^2 + (2\beta)x + (\gamma - \beta).$$

   Now a vector f in $\mathcal{P}_2(\mathbb{R})$ is a polynomial with degree at most 2 with real coefficients, and may be written in the form

$$f(x) = ax^2 + bx + c,$$

   where $a,\ b,\ c \in \mathbb{R}$.

   Setting

$$\beta = \frac{b}{2},\ \gamma = c + \frac{b}{2},\ \text{and}\ \alpha = a - c - \frac{b}{2},$$

   we see that

$$\begin{aligned}(a - c - \frac{b}{2})x^2 + (\frac{b}{2})(2x-1) + (c + \frac{b}{2})(x^2+1) \quad &= \quad ax^2 - cx^2 - \frac{b}{2}x^2 + bx - \frac{b}{2} + cx^2 + \frac{b}{2}x^2 + c + \frac{b}{2} \\ &= \quad ax^2 + bx + c \\ &= \quad f(x),\end{aligned}$$

   so that every vector in $\mathcal{P}_2(\mathbb{R})$ is a linear combination of vectors in the list $(x^2,\ 2x - 1,\ x^2+1)$.

   (b) Are the vectors independent in $\mathcal{P}_2(\mathbb{R})$? If so, show it; if not, provide a counterexample.

   Solution: If α, β, and γ are numbers so that

$$\alpha x^2 + 2\beta x - \beta + \gamma x^2 + \gamma = \mathbf{0},$$

   then by equating powers of x, we must have

$$\begin{aligned}\alpha + \gamma \quad &= \quad 0 \\ 2\beta \quad &= \quad 0 \\ -\beta + \gamma \quad &= \quad 0.\end{aligned}$$

   This implies that $\alpha = \beta = \gamma = 0$, so that the vectors are indeed independent.

2. Consider the vectors

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 3 \\ 3 & 9 \end{pmatrix},\ \text{and}\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   in $\mathcal{M}_2(\mathbb{C})$.

(a) Do the vectors span $\mathcal{M}_2(\mathbb{C})$? If so, show it; if not, provide an example of a vector not in their span.

Solution: I claim that the vector

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not in the span of the vectors above. To be certain, we check

$$\begin{aligned} \alpha \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 3 \\ 3 & 9 \end{pmatrix} + \gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2\alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 3\beta \\ 3\beta & 9\beta \end{pmatrix} + \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

To make things easier on ourselves, we rewrite the last equation as the *system*

$$\begin{aligned} 2\alpha - \gamma &= 0 \\ \alpha + 3\beta &= 1 \\ \alpha + 3\beta &= 0 \\ \alpha + 9\beta + \gamma &= 0, \end{aligned}$$

whose augmented matrix is given by

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 3 & 0 & | & 1 \\ 1 & 3 & 0 & | & 0 \\ 1 & 9 & 1 & | & 0 \end{pmatrix};$$

row reducing, we see that

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 3 & 0 & | & 1 \\ 1 & 3 & 0 & | & 0 \\ 1 & 9 & 1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 3 & 1/2 & | & 1 \\ 0 & 0 & 0 & | & -1 \\ 0 & 6 & 1 & | & -1 \end{pmatrix},$$

which is clearly an inconsistent system. Thus the list of vectors does not span $\mathcal{M}_2(\mathbb{C})$.

(b) Are the vectors independent in $\mathcal{M}_2(\mathbb{C})$? If so, show it; if not, provide a counterexample.

Solution: We need to check the equation

$$\begin{aligned} \alpha \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 3 \\ 3 & 9 \end{pmatrix} + \gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2\alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 3\beta \\ 3\beta & 9\beta \end{pmatrix} + \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which correspondes to the system *system*

$$\begin{aligned} 2\alpha - \gamma &= 0 \\ \alpha + 3\beta &= 0 \\ \alpha + 3\beta &= 0 \\ \alpha + 9\beta + \gamma &= 0. \end{aligned}$$

The augmented matrix for this system is

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 3 & 0 & | & 0 \\ 1 & 3 & 0 & | & 0 \\ 1 & 9 & 1 & | & 0 \end{pmatrix},$$

which row reduces to

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 3 & 0 & | & 0 \\ 1 & 3 & 0 & | & 0 \\ 1 & 9 & 1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 3 & 1/2 & | & 0 \\ 0 & 3 & 1/2 & | & 0 \\ 0 & 9 & 3/2 & | & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 1/6 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix},$$

which has nonzero solutions; e.g. $\gamma = 6$, $\beta = -1$, $\alpha = 3$. Thus the vectors are not independent.

3. Consider the vectors

$$\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in $\mathfrak{sl}(2,\mathbb{R})$, the vector space of all 2×2 trace 0 matrices over $\mathbb{R}$.

(a) Use a system of linear equations and Gauss-Jordan elimination on the resulting augmented matrix to show that the vectors span $\mathfrak{sl}(2,\mathbb{R})$.

Solution: Any matrix in $\mathfrak{sl}(2,\mathbb{R})$ can be written in the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

We wish to show that any matrix in $\mathfrak{sl}(2,\mathbb{R})$ may *also* be written in the form

$$\alpha \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix} + \delta \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2\alpha - \gamma + \delta & \alpha + \delta \\ \beta + 4\gamma + \delta & -2\alpha + \gamma - \delta \end{pmatrix}.$$

Thus we have the system of equations

$$\begin{aligned} 2\alpha - \gamma + \delta &= a \\ \alpha + \delta &= b \\ \beta + 4\gamma + \delta &= c \end{aligned}$$

(we may ignore the last equation, since it is a scalar multiple of the first). The matrix equation for this system is

$$\begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We may apply Gauss-Jordan elimination to the resulting augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 1 & a \\ 1 & 0 & 0 & 1 & b \\ 0 & 1 & 4 & 1 & c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & a/2 \\ 1 & 0 & 0 & 1 & b \\ 0 & 1 & 4 & 1 & c \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & a/2 \\ 0 & 0 & 1/2 & 1/2 & b - a/2 \\ 0 & 1 & 4 & 1 & c \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & a/2 \\ 0 & 1 & 4 & 1 & c \\ 0 & 0 & 1/2 & 1/2 & b - a/2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & a/2 \\ 0 & 1 & 4 & 1 & c \\ 0 & 0 & 1 & 1 & 2b - a \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & a/2 \\ 0 & 1 & 0 & -3 & c - 8b + 4a \\ 0 & 0 & 1 & 1 & 2b - a \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & b \\ 0 & 1 & 0 & -3 & c - 8b + 4a \\ 0 & 0 & 1 & 1 & 2b - a \end{array}\right)$$

This system is clearly consistent, so that the listed vectors span $\mathfrak{sl}(2, \mathbb{R})$.

(b) Use the equivalent conditions from Unit 1, Section 10 to show that the list is dependent.
Solution: If

$$\begin{pmatrix} 2\alpha - \gamma + \delta & \alpha + \delta \\ \beta + 4\gamma + \delta & -2\alpha + \gamma - \delta. \end{pmatrix} = \mathbf{0},$$

then the resulting matrix equation is

$$\begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \\ -2 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ -a \end{pmatrix}.$$

The coefficent matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \\ -2 & 0 & 1 & -1 \end{pmatrix}$$

is clearly determinant 0 (since the last row is a multiple of the first), so that the system

$$\alpha \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix} + \delta \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has nontrivial solutions.

4. In the last homework, we saw that if A is an $n \times n$ matrix and $\lambda \in \mathbb{F}$, then the set of all vectors x so that

$$Ax = \lambda x$$

is a subspace of $\mathbb{F}^n$.

Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\lambda = 1$.

(a) Let E be the subspace of $\mathbb{R}^3$ of all vectors x so that

$$Ax = \lambda x.$$

Find a parametric description for a general vector in E.
Solution: Since $\lambda = 1$, we are looking for vectors $x \in \mathbb{R}^3$ so that

$$Ax = x, \text{ or equivalently } (A - I)x = \mathbf{0}.$$

Using the second equation above, we wish to find $x \in \mathbb{R}^3$ so that

$$\begin{aligned} \mathbf{0} &= (A - I)x \\ &= \begin{pmatrix} 1-1 & -1 & 1 \\ 0 & 2-1 & -1 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \end{aligned}$$

Reducing the augmented matrix for the system, we have

$$\begin{pmatrix} 0 & -1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus *both* x_1 and x_3 are free variables; parameterizing

$$x_1 = t, \; x_3 = s,$$

we have

$$x = \begin{pmatrix} t \\ s \\ s \end{pmatrix}.$$

Thus E is the subspace of all vectors in $\mathbb{R}^3$ of the form

$$\begin{pmatrix} t \\ s \\ s \end{pmatrix},$$

$s, t \in \mathbb{R}$.

(b) Find a list of two vectors that spans the subspace E.
Example: Every vector in E may be written in the form

$$\begin{aligned} \begin{pmatrix} t \\ s \\ s \end{pmatrix} &= \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ s \end{pmatrix} \\ &= t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus the list

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

spans E.

5. Recall that $\mathbb{C}^3$, the vector space of all 3×1 matrices with complex entries, is a vector space over $\mathbb{C}$, with basis

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

(a) Show that the list of vectors above is *not* a basis for $\mathbb{C}^3$ when $\mathbb{C}^3$ is viewed as a vector space over $\mathbb{R}$.

Solution: Clearly there are no real numbers so that

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix},$$

so the vectors *do not* span $\mathbb{C}^3$ over $\mathbb{R}$.

(b) Extend the list of vectors above to a basis for $\mathbb{C}^3$ over $\mathbb{R}$.

Example: It seems reasonable to guess that the list

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \right)$$

could be a basis for $\mathbb{C}^3$ over $\mathbb{R}$.

Let's verify that the vectors span $\mathbb{C}^3$: every vector in $\mathbb{C}^3$ can be written in the form

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$

where α, β, $\gamma \in \mathbb{C}$; however, rewriting

$$\begin{aligned} \alpha &= a_1 + a_2 i \\ \beta &= b_1 + b_2 i \\ \gamma &= c_1 + c_2 i, \end{aligned}$$

where a_j, b_j, $c_j \in \mathbb{R}$, we see that

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &= \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + i \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \\ &= a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}. \end{aligned}$$

Thus the list

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \right)$$

spans $\mathbb{C}^3$.

To show that the vectors are independent, we need merely observe that for a_1, $a_2 \in \mathbb{R}$,

$$a_1 + a_2 i = 0 \iff a_1 = a_2 = 0.$$

Thus

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if each a_j, b_j, c_j above is identically 0.

6. Recall that an $n \times n$ matrix A is *skew-symmetric* if $A^\top = -A$. It is easy to see that the set $\mathfrak{so}(n, \mathbb{F})$ of all $n \times n$ skew-symmetric matrices with entries in $\mathbb{F}$ is a subspace of $\mathcal{M}_n(\mathbb{F})$.

   (a) Find a basis for $\mathfrak{so}(2, \mathbb{C})$.
   Example: One possible choice of basis is

$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

   (b) Find a basis for $\mathfrak{so}(3, \mathbb{C})$; prove that your list of vectors forms a basis.
   Example: Any matrix in $\mathfrak{so}(3, \mathbb{C})$ can be written in the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

   Thus I claim that the list

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right)$$

   is a basis.
   Now given any 3×3 skew symmetric matrix

$$u = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

   it is clear that u is a linear combination of the vectors in our list:

$$a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

   The vectors in our list are also clearly independent, since they do not have any interacting nonzero entries.

   (c) Find a formula for the number of vectors in a basis for $\mathfrak{so}(n, \mathbb{C})$.
   Solution: To find the number of vectors in a basis for $\mathfrak{so}(n, \mathbb{C})$, we simply need to count the number of *above diagonal entries* in an $n \times n$ matrix.

Row 1 of our matrix has n entries, thus $n-1$ above diagonal entries. Row 2 has $n-2$ above diagonal entries; etc. So the total number of above diagonal entries in an $n \times n$ matrix is the sum of the first $n-1$ numbers, which can be written as

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}.$$

7. Suppose that the list

$$(v_1,\ v_2,\ \ldots,\ v_m)$$

is linearly independent, and that w is a vector in V so that that the list

$$(v_1 + w,\ v_2 + w,\ \ldots,\ v_m + w)$$

is linearly *dependent*. Prove that

$$w \in \operatorname{span}(v_1,\ v_2,\ \ldots,\ v_m).$$

Solution: If $(v_1 + w,\ v_2 + w,\ \ldots,\ v_m + w)$ is a dependent list, then there are constants α_1, $\ldots$, α_n, at least one $\alpha_i \neq 0$, so that

$$\alpha_1 v_1 + \alpha_1 w + \alpha_2 v_2 + \alpha_2 w + \ldots + \alpha_m v_m + \alpha_m w \quad = \quad \mathbf{0}.$$

Rearranging terms in the equation, we see that

$$w(\alpha_1 + \alpha_2 + \ldots + \alpha_m) = -\alpha_1 v_1 - \alpha_2 v_2 - \ldots - \alpha_m v_m.$$

Now we know that

$$-\alpha_1 v_1 - \alpha_2 v_2 - \ldots - \alpha_m v_m \neq \mathbf{0}$$

since at least one $\alpha_i \neq 0$ and the v_i are independent. Thus

$$w(\alpha_1 + \alpha_2 + \ldots + \alpha_m) \neq \mathbf{0},$$

which means that

$$w \neq \mathbf{0} \text{ and } \alpha_1 + \alpha_2 + \ldots + \alpha_m \neq 0.$$

Thus we can divide by $\alpha_1 + \alpha_2 + \ldots + \alpha_m$, so that w may be written as the linear combination

$$w = \frac{-\alpha_1}{\alpha_1 + \alpha_2 + \ldots + \alpha_m} v_1 + \frac{-\alpha_2}{\alpha_1 + \alpha_2 + \ldots + \alpha_m} v_2 + \ldots + \frac{-\alpha_m}{\alpha_1 + \alpha_2 + \ldots + \alpha_m} v_m.$$

8. Prove that a vector space V is infinite dimensional if and only if there is an infinite sequence

$$v_1,\ v_2,\ \ldots$$

of vectors in V so that the list

$$(v_1,\ v_2,\ \ldots,\ v_m)$$

is independent for every positive integer m.

Solution: "$\Leftarrow$" If there is an infinite sequence

$$v_1,\ v_2,\ \ldots$$

of vectors in V so that the list

$$(v_1,\ v_2,\ \ldots,\ v_m)$$

is independent for every positive integer m, then since *every* spanning list is at least as long as every independent list, there is no finite spanning list for V; thus V is infinite dimensional.

"$\Rightarrow$" If V is infinite dimensional, it has no finite spanning list. We proceed by contradiction: suppose that there *is no such sequence of vectors*. Then for every infinite sequence

$$v_1,\ v_2,\ \ldots$$

of vectors, there is some finite list

$$(v_1,\ v_2,\ \ldots,\ v_m)$$

that is dependent. This implies that for *any* finite list of independent vectors in V, say

$$S = (u_1,\ u_2,\ \ldots,\ u_m),$$

there are only finitely many vectors in V that may be added to S while maintaining independence of S. Without loss of generality we may assume that S is independent, and that *no* vector in V may be added to S while maintaining independence. Thus *every* vector $u \in V$ is a linear combination of the vectors in the (finite) list S, a contradiction.