1. Consider the vectors x^2, $2x-1$, and x^2+1 in $\mathcal{P}_2(\mathbb{R})$.

   (a) Do the vectors span $\mathcal{P}_2(\mathbb{R})$? If so, show it; if not, provide an example of a vector not in their span.

   (b) Are the vectors independent in $\mathcal{P}_2(\mathbb{R})$? If so, show it; if not, provide a counterexample.

2. Consider the vectors
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 9 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
in $\mathcal{M}_2(\mathbb{C})$.

   (a) Do the vectors span $\mathcal{M}_2(\mathbb{C})$? If so, show it; if not, provide an example of a vector not in their span.

   (b) Are the vectors independent in $\mathcal{M}_2(\mathbb{C})$? If so, show it; if not, provide a counterexample.

3. Consider the vectors
$$\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
in $\mathfrak{sl}(2, \mathbb{R})$, the vector space of all 2×2 trace 0 matrices over $\mathbb{R}$.

   (a) Use a system of linear equations and Gauss-Jordan elimination on the resulting augmented matrix to show that the vectors span $\mathfrak{sl}(2, \mathbb{R})$.

   (b) Use the equivalent conditions from Unit 1, Section 10 to show that the list is dependent.

4. In the last homework, we saw that if A is an $n \times n$ matrix and $\lambda \in \mathbb{F}$, then the set of all vectors x so that
$$Ax = \lambda x$$
is a subspace of $\mathbb{F}^n$.

   Let
$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
and $\lambda = 1$.

   (a) Let E be the subspace of $\mathbb{R}^3$ of all vectors x so that
$$Ax = \lambda x.$$
Find a parametric description for a general vector in E.

   (b) Find a list of two vectors that spans the subspace E.

5. Recall that $\mathbb{C}^3$, the vector space of all 3×1 matrices with complex entries, is a vector space over $\mathbb{C}$, with basis
$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

(a) Show that the list of vectors above is *not* a basis for $\mathbb{C}^3$ when $\mathbb{C}^3$ is viewed as a vector space over $\mathbb{R}$.

(b) Extend the list of vectors above to a basis for $\mathbb{C}^3$ over $\mathbb{R}$.

6. Recall that an $n \times n$ matrix A is *skew-symmetric* if $A^\top = -A$. It is easy to see that the set $\mathfrak{so}(n, \mathbb{F})$ of all $n \times n$ skew-symmetric matrices with entries in $\mathbb{F}$ is a subspace of $\mathcal{M}_n(\mathbb{F})$.

   (a) Find a basis for $\mathfrak{so}(2, \mathbb{C})$.

   (b) Find a basis for $\mathfrak{so}(3, \mathbb{C})$; prove that your list of vectors forms a basis.

   (c) Find a formula for the number of vectors in a basis for $\mathfrak{so}(n, \mathbb{C})$.

7. Suppose that the list
$$(v_1,\ v_2,\ \dots,\ v_m)$$
is linearly independent, and that w is a vector in V so that that the list
$$(v_1 + w,\ v_2 + w,\ \dots,\ v_m + w)$$
is linearly *dependent*. Prove that
$$w \in \operatorname{span}(v_1,\ v_2,\ \dots,\ v_m).$$

8. Prove that a vector space V is infinite dimensional if and only if there is an infinite sequence
$$v_1,\ v_2,\ \dots$$
of vectors in V so that the list
$$(v_1,\ v_2,\ \dots,\ v_m)$$
is independent for every positive integer m.