Recall that a Lie algebra is a vector space L over a field with an additional operation called the bracket, denoted by $[\cdot,\cdot]$,

$$[x, y] \in L \ \forall \ x, \ y \in L.$$

The bracket operation must have the following properties:

1. For all x, y, x_1, x_2, y_1, $y_2 \in L$, all $\alpha \in \mathbb{F}$,

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y],$$

$$[x, y_1 + y_2] = [x, y_1] + [x, y_2],$$

   and

$$[\alpha x, y] = [x, \alpha y] = \alpha[x, y].$$

   This is referred to as the *bilinear* property.

2. For all $x \in L$, $[x, x] = 0$.

3. For all x, y, and $z \in L$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \mathbf{0}$$

   (Jacobi identity).

Consider the vector space $\mathfrak{gl}(2, \mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ (we have used the notation $\mathcal{M}_2(\mathbb{R})$ for this space, but the notation $\mathfrak{gl}(2, \mathbb{R})$ is conventional in Lie theory). Define

$$[\cdot,\cdot] : \mathfrak{gl} \times \mathfrak{gl} \to \mathfrak{gl}$$

by

$$[x, y] = xy - yx$$

(notice that this is a *different* definition from the bracket on $\mathbb{R}^3$!) We wish to show that the vector space $\mathfrak{gl}(2, \mathbb{R})$, together with the operation

$$[x, y] = xy - yx,$$

is a Lie algebra.

1. Explain why $\mathfrak{gl}(2, \mathbb{R})$ is closed under the $[\cdot,\cdot]$.
2. Show that the operation is bilinear.
3. Show that $[x, x] = \mathbf{0}$ for all $x \in \mathfrak{gl}(2, \mathbb{R})$.
4. Show that the operation satisfies the Jacobi identity.

Again, the above statements show that $(\mathfrak{gl}(2, \mathbb{R}), [\cdot,\cdot])$ is a Lie algebra.