1. It is clear from the definition that \( \mathbb{R}^3 \) is closed under the operation; indeed, \([x, y] = x \times y\) is a \(3 \times 1\) matrix with real entries. Thus

\[
[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}.
\]

2. Under addition in the first coordinate, we have

\[
[x + y, z] = (x + y) \times z
\]

\[
= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}
\]

\[
= \begin{pmatrix} (x_1 + y_1)z_3 - (x_3 + y_3)z_1 \\ (x_3 + y_3)z_1 - (x_1 + y_1)z_3 \\ (x_1 + y_1)z_2 - (x_2 + y_2)z_1 \end{pmatrix}
\]

\[
= \begin{pmatrix} x_1z_3 - x_3z_2 \\ x_3z_1 - x_1z_3 \\ x_1z_2 - x_2z_1 \end{pmatrix} + \begin{pmatrix} y_1z_3 - y_3z_2 \\ y_3z_1 - y_1z_3 \\ y_1z_2 - y_2z_1 \end{pmatrix}
\]

\[
= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}
\]

\[
= [x, z] + [y, z].
\]

Thus the bracket distributes over addition in the first coordinate; similarly, it distributes over addition in the second coordinate. It is also easy to see that constants factor out of coordinates of the bracket, so that the bracket is bilinear.

3. We can see that \([x, x] = 0\) using geometric reasoning: the formula

\[
|a \times b| = |a||b| \sin \theta
\]

relates the length of the cross product of a pair of vectors to the sine of the angle between them. In our case with \(a = b = x\), \(\theta = 0\), so that

\[
|x \times x| = |x|^2 \sin 0
\]

\[
= 0.
\]

Now the only length 0 vector in \(\mathbb{R}^3\) is \(0\), so that

\[
[x, x] = x \times x = 0.
\]
Algebraically, we have

\[
[x, x] = \begin{pmatrix}
x_2x_3 - x_3x_2 \\
x_3x_1 - x_1x_3 \\
x_1x_2 - x_2x_1
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
= 0.
\]

4. We need to evaluate

\[[[x, y], z] + [[y, z], x] + [[z, x], y].\]

Accordingly, we have

\[
[[x, y], z] + [[y, z], x] + [[z, x], y] = [x \times y, z] + [y \times z, x] + [z \times x, y]
= \begin{pmatrix}
x_2y_2z_1 - x_1y_2z_2 + x_3y_1z_3 - x_1y_3z_3 \\
-x_2y_1z_1 + x_1y_2z_1 + x_3y_2z_3 - x_2y_3z_3 \\
-x_3y_1z_1 + x_1y_3z_1 - x_3y_2z_2 + x_2y_3z_2
\end{pmatrix}
+ \begin{pmatrix}
x_2y_2z_3 + x_3y_3z_1 - x_2y_1z_2 - x_3y_1z_3 \\
-x_1y_2z_1 + x_1y_1z_2 + x_3y_1z_2 - x_3y_2z_3 \\
-x_1y_3z_1 - x_2y_3z_2 + x_1y_1z_3 + x_2y_2z_3
\end{pmatrix}
+ \begin{pmatrix}
-x_2y_2z_1 - x_3y_2z_1 + x_1y_2z_2 + x_1y_3z_3 \\
x_2y_2z_1 - x_1y_2z_2 - x_3y_2z_2 + x_2y_3z_3 \\
x_3y_1z_1 + x_3y_2z_2 - x_1y_1z_3 - x_2y_2z_3
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Thus the bracket satisfies the Jacobi identity.