1. It is clear from the definition that $\mathbb{R}^3$ is closed under the operation; indeed, $[x, y] = x \times y$ is a 3×1 matrix with real entries. Thus

$$[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}.$$

2. Under addition in the first coordinate, we have

$$\begin{aligned}
[x+y, z] &= (x+y) \times z \\
&= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\
&= \begin{pmatrix} (x_2 + y_2)z_3 - (x_3 + y_3)z_2 \\ (x_3 + y_3)z_1 - (x_1 + y_1)z_3 \\ (x_1 + y_1)z_2 - (x_2 + y_2)z_1 \end{pmatrix} \\
&= \begin{pmatrix} x_2 z_3 - x_3 z_2 \\ x_3 z_1 - x_1 z_3 \\ x_1 z_2 - x_2 z_1 \end{pmatrix} + \begin{pmatrix} y_2 z_3 - y_3 z_2 \\ y_3 z_1 - y_1 z_3 \\ y_1 z_2 - y_2 z_1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\
&= [x, z] + [y, z].
\end{aligned}$$

Thus the bracket distributes over addition in the first coordinate; similarly, it distributes over addition in the second coordinate. It is also easy to see that constants factor out of coordinates of the bracket, so that the bracket is bilinear.

3. We can see that $[x, x] = \mathbf{0}$ using geometric reasoning: the formula

$$|a \times b| = |a||b| \sin\theta$$

relates the length of the cross product of a pair of vectors to the sine of the angle between them. In our case with $a = b = x$, $\theta = 0$, so that

$$\begin{aligned}
|x \times x| &= |x|^2 \sin 0 \\
&= 0.
\end{aligned}$$

Now the only length 0 vector in $\mathbb{R}^3$ *is* $\mathbf{0}$, so that

$$[x, x] = x \times x = \mathbf{0}.$$

Algebraically, we have

$$\begin{aligned}
[x, x] &= \begin{pmatrix} x_2x_3 - x_3x_2 \\ x_3x_1 - x_1x_3 \\ x_1x_2 - x_2x_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}$$

4. We need to evaluate

$$[[x, y], z] + [[y, z], x] + [[z, x], y].$$

Accordingly, we have

$$\begin{aligned}
[[x, y], z] + [[y, z], x] + [[z, x], y] &= [x \times y, z] + [y \times z, x] + [z \times x, y] \\
&= \begin{pmatrix} x_2y_1z_2 - x_1y_2z_2 + x_3y_1z_3 - x_1y_3z_3 \\ -x_2y_1z_1 + x_1y_2z_1 + x_3y_2z_3 - x_2y_3z_3 \\ -x_3y_1z_1 + x_1y_3z_1 - x_3y_2z_2 + x_2y_3z_2 \end{pmatrix} \\
&+ \begin{pmatrix} x_2y_2z_1 + x_3y_3z_1 - x_2y_1z_2 - x_3y_1z_3 \\ -x_1y_2z_1 + x_1y_1z_2 + x_3y_3z_2 - x_3y_2z_3 \\ -x_1y_3z_1 - x_2y_3z_2 + x_1y_1z_3 + x_2y_2z_3 \end{pmatrix} \\
&+ \begin{pmatrix} -x_2y_2z_1 - x_3y_3z_1 + x_1y_2z_2 + x_1y_3z_3 \\ x_2y_1z_1 - x_1y_1z_2 - x_3y_3z_2 + x_2y_3z_3 \\ x_3y_1z_1 + x_3y_2z_2 - x_1y_1z_3 - x_2y_2z_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus the bracket satisfies the Jacobi identity.