Vector spaces are our topic of interest in this course, but it turns out that there are many other types of "spaces" with interesting properties. One of these is the *Lie algebra* (pronounced "lee"). A Lie algebra is, first of all, a plain vanilla vector space L over a field, so that we have the operations of vector addition and scalar multiplication on L. In addition to these operations, a Lie algebra must have an additional operation called the bracket, denoted by $[\cdot,\cdot]$,

$$[x, y] \in L \ \forall \ x, \ y \in L.$$

Of course, this new operation must behave nicely; to be a Lie algebra, we require that the bracket obey the following rules:

1. For all x, y, x_1, x_2, y_1, $y_2 \in L$, all $\alpha \in \mathbb{F}$,

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y],$$

$$[x, y_1 + y_2] = [x, y_1] + [x, y_2],$$

   and

$$[\alpha x, y] = [x, \alpha y] = \alpha[x, y].$$

   This is referred to as the *bilinear* property.

2. For all $x \in L$, $[x, x] = 0$.

3. For all x, y, and $z \in L$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \mathbf{0}$$

   (recall that this property is known as the Jacobi identity).

Over the next few challenge problems, we will be looking at examples of different Lie algebras. Keep in mind that if L is a Lie algebra, then L is already a vector space, and thus has the operations of vector addition and scalar multiplication. The new operation $[\cdot,\cdot]$ sets it apart from structures that are *only* vector spaces.

Consider $\mathbb{R}^3$ as a vector space over $\mathbb{R}$, and recall that the *cross-product* of vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

in $\mathbb{R}^3$ is defined by

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_1 y_3 - x_3 y_1 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

We are going to show that the vector space $\mathbb{R}^3$, together with the operation

$$[x, y] = x \times y,$$

is a Lie algebra.

1. Show that $\mathbb{R}^3$ is closed under the $[\cdot,\cdot]$.

2. Show that the operation is bilinear.
3. Show that $[x, x] = \mathbf{0}$ for all $x \in \mathbb{R}^3$.
4. Show that the operation satisfies the Jacobi identity.

Once you have proved the above statements, you have *also* proved that $(\mathbb{R}, [\cdot, \cdot])$ is a Lie algebra.