1. Since the bracket on $\mathcal{M}_n(\mathbb{F})$ is defined by

$$[X,Y] = XY - YX,$$

is is clear that

$$\begin{aligned} [X,X] &= XX - XX \\ &= \mathbf{0}. \end{aligned}$$

2. (a)

$$\begin{aligned} [[X,Y],Z] &= [X,Y]Z - Z[X,Y] \\ &= \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(b)

$$\begin{aligned} [[U,Z],X] &= [Y,Z]X - X[Y,Z] \\ &= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} [[Z,X],Y] &= [Z,X]Y - Y[Z,X] \\ &= \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ -2 & -2 \end{pmatrix}. \end{aligned}$$

(d)

$$\begin{aligned} [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] &= \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

3. In general,

$$\begin{aligned} [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] &= [XY - YX, Z] + [YZ - ZY, X] + [ZX - XZ, Y] \\ &= (XY - YX)Z - Z(XY - YX) + (YZ - ZY)X \\ &\quad -X(YZ - ZY) + (ZX - XZ)Y - Y(ZX - XZ) \\ &= XYZ - YXZ - ZXY + ZYX + YZX - ZYX - XYZ \\ &\quad +XZY + ZXY - XZY - YZX + YXZ \\ &= \mathbf{0}. \end{aligned}$$