Coordinates

As indicated in the previous section, bases are important because they give us a precise way to describe the geometry of their ambient vector spaces. To make this idea more concrete, and to give us a way to use matrix arithmetic to encode and understand our vector spaces, we introduce the idea of *coordinate systems*:

Definition. If $S = (v_1,\ v_2,\ \ldots,\ v_n)$ is a basis for the vector space V and the vector v in V is the linear combination

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n,$$

then the scalars α_1, α_2, $\ldots$, α_n are called the *coordinates* of v relative to the basis S, and the vector

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

in $\mathbb{F}^n$ is called the *coordinate vector* of v relative to the basis S, denoted by

$$(v)_S = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Before we engage in an in-depth discussion of the ideas presented in the definition, let's apply it to the example in $\mathbb{R}^2$ that we have now seen multiple times: we have seen that the lists

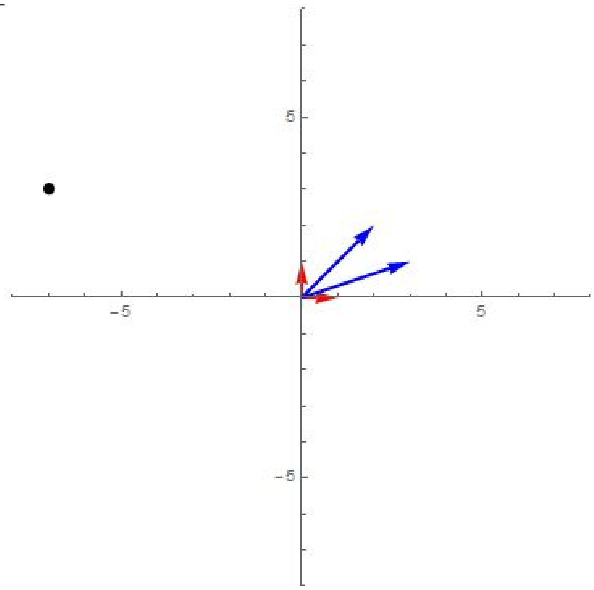
$$B_1 = (e_1,\ e_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

and

$$B_2 = (v_1,\ v_2) = \left(\begin{pmatrix} 3 \\ 1 \end{pmatrix},\ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$$

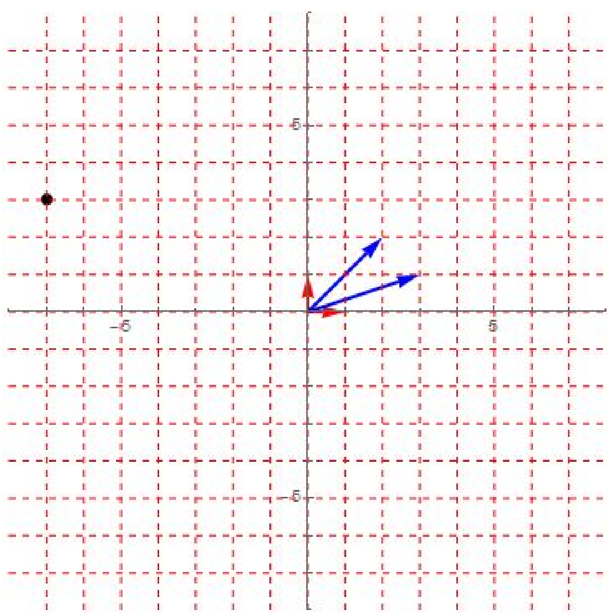
both form bases for $\mathbb{R}^2$.

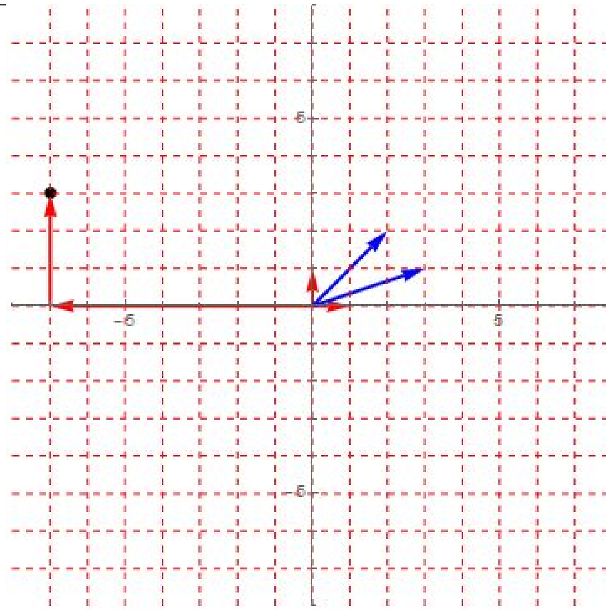
Let's pick a vector (or point) p in $\mathbb{R}^2$ and compare its coordinates in the two different systems:



We'll start by writing the coordinates of this point in terms of the first basis

$$B_1 = (e_1,\ e_2) :$$





From the origin, we have to move $-7e_1$ and $3e_2$ to get to p, so we think of p as the linear combination

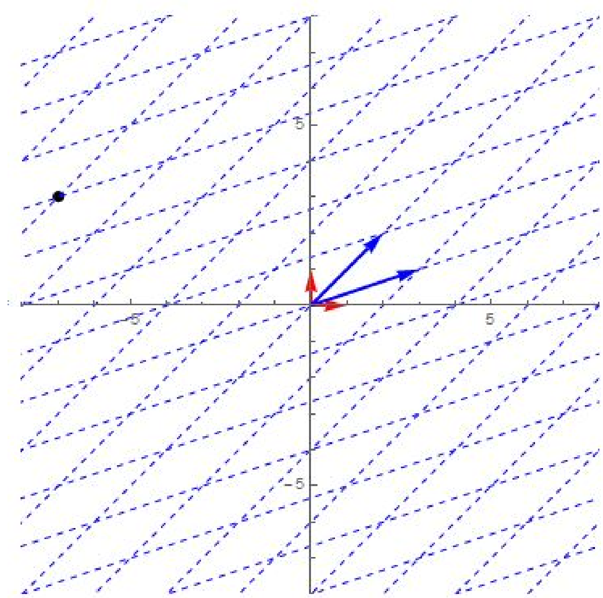
$$p = -7e_1 + 3e_2;$$

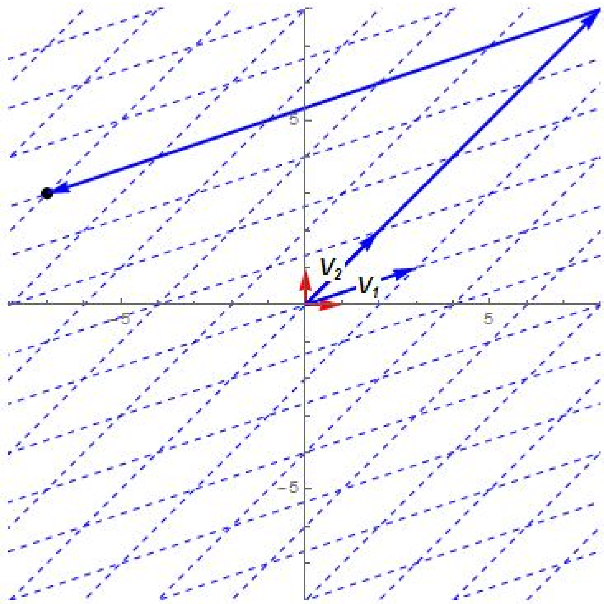
so with this choice of basis, the coordinates for p are given by the vector

$$(p)_{B_1} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}.$$

Now let's write the coordinates of p in terms of the second basis

$$B_2 = (v_1,\ v_2):$$





In order to get to p, we need to move from the origin by $-5v_1$ and $4v_2$; so with B_2 as our chosen basis, p is the linear combination

$$p = -5v_1 + 4v_2,$$

and has coordinates given by the vector

$$(p)_{B_2} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

In other words, I have options as to how I describe the location of p: if I like the first basis, I can refer to the location of p as $(-7, 3)$. If, however, I prefer the second basis, then I can refer to p's location as $(-5, 4)$. As long as I specify which basis I'm using to mark the location, there is no ambiguity as to the location of the point.

Key Point. Coordinates in a vector space are just a specific way to refer to the location of vectors relative to a chosen basis; a vector's coordinates will look different if we use a different basis to describe locations, but the coordinates *will* refer to the same vector.

Example 1

The matrix

$$v = \begin{pmatrix} -4 & 2 \\ 0 & -3 \end{pmatrix}$$

is a vector in the vector space $\mathcal{U}_2(\mathbb{R})$ of all real upper triangular 2×2 matrices. Find the coordinates of v_1 relative to the basis

$$B = \left(e_{11},\ e_{12},\ e_{22} \right),$$

where e_{ij} is the 2×2 matrix with a 1 in the i, j entry and 0s elsewhere.

To find the coordinates of v, we first need to write v as a linear combination of the basis vectors. Of course, it is quite easy to see that the correct linear combination is

$$v = -4e_{11} + 2e_{12} - 3e_{22};$$

so the coordinates of v are given by the vector

$$(v)_B = \begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix}.$$

Example 2

The vectors $f_1(x) = 2x - 3$, $f_2(x) = x^2 + 1$, and $f_3(x) = 2x^2 - x$ are linearly independent and span the vector space $\mathcal{P}_2(\mathbb{R})$ of all real-valued polynomials of degree no more than 2, so that the list

$$B = (f_1,\ f_2,\ f_3)$$

is a basis for $\mathcal{P}_2(\mathbb{R})$. Find the coordinates for the vector

$$g(x) = 3x^2 + 4x - 10$$

relative to this basis.

We need to write g as a linear combination of f_1, f_2, and f_3; in other words, we need scalars a, b, and c so that

$$g = af_1 + bf_2 + cf_3;$$

these scalars are precisely the coordinates of g relative to this basis.

Let's make the calculations: we want

$$\begin{aligned} 3x^2 + 4x - 10 &= af_1 + bf_2 + cf_3 \\ &= a(2x-3) + b(x^2+1) + c(2x^2 - x) \\ &= 2ax - 3a + bx^2 + b + 2cx^2 - cx \\ &= (b+2c)x^2 + (2a-c)x + (-3a+b). \end{aligned}$$

So we need to solve the system of equations

$$\begin{aligned} 3 &= b + 2c \\ 4 &= 2a - c \\ -10 &= -3a + b. \end{aligned}$$

This corresponds to the augmented matrix

$$\begin{pmatrix} 0 & 1 & 2 & | & 3 \\ 2 & 0 & -1 & | & 4 \\ -3 & 1 & 0 & | & -10 \end{pmatrix};$$

we apply elementary row operations to find the solution to the system:

$$\begin{pmatrix} 0 & 1 & 2 & | & 3 \\ 2 & 0 & -1 & | & 4 \\ -3 & 1 & 0 & | & -10 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & -1 & | & 4 \\ 0 & 1 & 2 & | & 3 \\ -3 & 1 & 0 & | & -10 \end{pmatrix}$$

$$\text{divide row 1 by 2} \quad \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & | & 2 \\ 0 & 1 & 2 & | & 3 \\ -3 & 1 & 0 & | & -10 \end{pmatrix}$$

$$\text{add 3 times row 1 to row 3} \quad \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & | & 2 \\ 0 & 1 & 2 & | & 3 \\ 0 & 1 & -\frac{3}{2} & | & -4 \end{pmatrix}$$

$$\text{add -1 times row 2 to row 3} \quad \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & | & 2 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & -\frac{7}{2} & | & -7 \end{pmatrix}$$

$$\text{multiply row 3 by -2/7} \quad \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & | & 2 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$\text{add -2 row 3 to row 2} \quad \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

$$\text{ad 1/2 row 3 to row 1} \quad \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}.$$

So we see that

$$a = 3, \; b = -1, \; \text{and } c = 2,$$

which means that

$$g = 3f_1 - f_2 + 2f_3;$$

thus the coordinates for g relative to the basis B are give by

$$(g)_B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$