Subspaces

In the previous section, we saw that the set $\mathcal{U}_2(\mathbb{R})$ of all real upper triangular 2×2 matrices, i.e. the set of all matrices of the form

$$\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix},$$

together with the usual operations of matrix addition and scalar multiplication, is a vector space.

You may have noticed that $\mathcal{U}_2(\mathbb{R})$ looks a good bit like another vector space we have already studied, specifically $\mathcal{M}_2(\mathbb{R})$, the space of all 2×2 real matrices. In fact, every vector in $\mathcal{U}_2(\mathbb{R})$ is *also* a vector in $\mathcal{M}_2(\mathbb{R})$ (although the reverse is not true–many 2×2 matrices are not upper triangular!), and the two vector spaces have the same operations of addition and scalar multiplication.

There is a sense in which the vector space $\mathcal{U}_2(\mathbb{R})$ is "living inside" the vector space $\mathcal{M}_2(\mathbb{R})$; this phenomenon is common enough that we will give it a name:

Definition 1.32. A subset U of a vector space V is called a *subspace* of V if it is a vector space in its own right, with the same addition and scalar multiplication defined in V.

Using our earlier example, we can now say that $\mathcal{U}_2(\mathbb{R})$ is a subspace of $\mathcal{M}_2(\mathbb{R})$.

Determining if Subsets are Subspaces

When we are attempting to determine if a set U is a vector space, there is a particular advantage to recognizing that U is a subset of another set V that is already known to be a vector space.

For example, let's go back to the set $\mathcal{U}_2(\mathbb{R})$ of all upper triangular 2×2 matrices, and the vector space $\mathcal{M}_2(\mathbb{R})$ of all 2×2 matrices. When we checked to see if $\mathcal{U}_2(\mathbb{R})$ itself is a vector space, we skipped several necessary steps, ignoring most of the vector space axioms. As a result, you may be a bit concerned that $\mathcal{U}_2(\mathbb{R})$ is not actually a vector space in its own right; perhaps it fails, say, the association axiom:

$$u + (v + w) = (u + v) + w.$$

Fortunately for us, we don't have to worry about this axiom: since $\mathcal{U}_2(\mathbb{R})$ is a subset of $\mathcal{M}_2(\mathbb{R})$, every vector in $\mathcal{U}_2(\mathbb{R})$ is also in $\mathcal{M}_2(\mathbb{R})$. Of course, we know that $\mathcal{M}_2(\mathbb{R})$ is a vector space, so the rule

$$u + (v + w) = (u + v) + w$$

works for *all* vectors in $\mathcal{M}_2(\mathbb{R})$–including all of the vectors in $\mathcal{U}_2(\mathbb{R})$. Thus we don't have to check axiom 4–since $\mathcal{U}_2(\mathbb{R})$ is a subset of $\mathcal{M}_2(\mathbb{R})$, its vectors "inherit" the behavior of the vectors in $\mathcal{M}_2(\mathbb{R})$.

Key Point. If we wish to determine if the set U forms a vector space with the operations of addition and scalar multiplication, and recognize that all of the vectors in U are also vectors in a (perhaps larger) set V known to be a vector space, then the vectors in U will automatically obey many of the vector space axioms. Thus we will not have to check every single axiom to determine whether or not U is a vector space.

The point above leads to a question: if we *do* recognize the set U as a subset of a known vector space V, what axioms do we need to check to determine whether or not U is a vector space? The following theorem answers this question:

Theorem 1.34. A subset U of the vectors of a vector space V is a subspace of V (and thus a vector space in its own right) if and only if the following conditions are satisfied:

1. **Additive Identity:** $\mathbf{0} \in U$
2. **Closure under Addition:** $u, v \in U \implies u + v \in U$
3. **Closure under Scalar Multiplication:** $\lambda \in \mathbb{F}, u \in U \implies \lambda u \in U$

Again, the theorem says that, if a subset U of a vector space V is nonempty and closed under the operations of addition and scalar multiplication, then we are guaranteed that U is a subspace of V, and thus a vector space itself.

Proof. $\implies$ If U is a subspace of V, then U is a vector space itself and clearly satisfies all of the properties.

$\impliedby$ On the other hand, we must show that, if U satisfies the conditions, then it satisfies *all* of the vector space axioms. We discuss each one below:

1. Closure under Addition: Follows by assumption.
2. Closure under Scalar Multiplication: Follows by assumption.
3. Commutativity: Inherited from V, since every vector in U is also in V, and the operations are identical.
4. Associativity: Inherited from V.
5. Additive Identity: Follows by assumption.
6. Additive Inverse: If $u \in U$, we must guarantee that $-u$ is as well. Since $(-1)u \in U$ by assumption and $(-1)u = -u$, additive inverses are always in U.
7. Multiplicative Identity: Inherited from V.
8. Distribution of Scalar Multiplication over Vector Addition: Inherited from V.
9. Distribution of Scalar Multiplication over Scalar Addition: Inherited from V.

Thus U satisfies all of the properties, and is a vector space in its own right; it is therefore a subspace of V.

Returning to the example above of the set $\mathcal{U}_2(\mathbb{R})$ of all real upper triangular 2×2 matrices, and the vector space $\mathcal{M}_2(\mathbb{R})$ of all real 2×2 matrices, it is now clear that $\mathcal{U}_2(\mathbb{R})$ is subspace of $\mathcal{M}_2(\mathbb{R})$–we know that:

1. The 2×2 zero matrix is upper triangular;

2. The sum of two upper triangular matrices is upper triangular, and that

3. The scalar product of an upper triangular matrix with a real number is upper triangular.

The theorem says that, since we know that $\mathcal{U}_2(\mathbb{R})$ is a subset of $\mathcal{M}_2(\mathbb{R})$, and that $\mathcal{M}_2(\mathbb{R})$ is itself a vector space, these are the only three conditions we need to check to be certain that $\mathcal{U}_2(\mathbb{R})$ is a vector space, as well.

Examples of Subsets that are Subspaces

Example 1

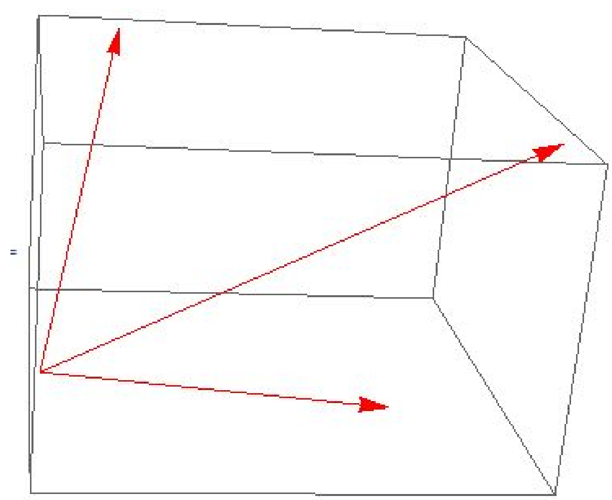
Let W be the set of all vectors in $\mathbb{R}^3$ of the form

$$\langle x, x+z, z \rangle.$$

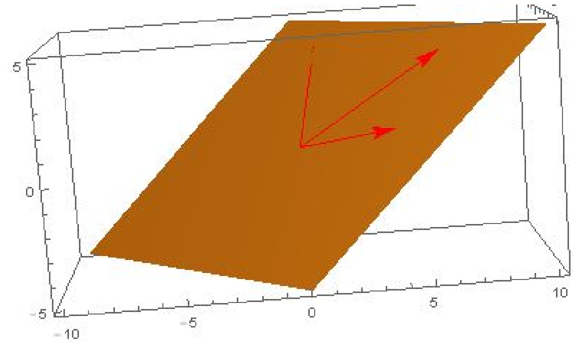
For example,

$$\langle 1, 4, 3 \rangle$$

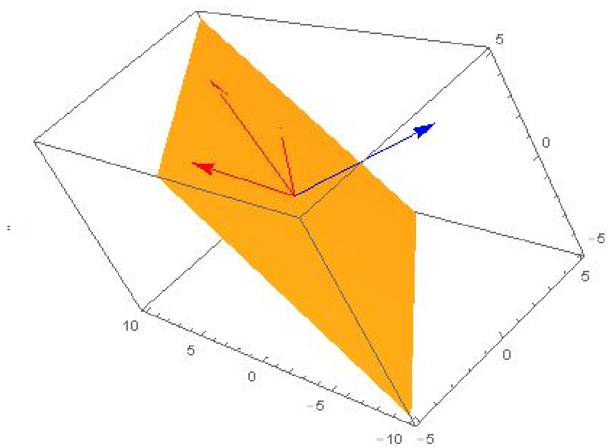
is a vector in W. Three such vectors are graphed below in $\mathbb{R}^3$:



All of the vectors from W "cover" the orange plane included in the graph below:



You can think of W as the yellow plane; clearly every vector in W is also in $\mathbb{R}^3$, but *not* every vector in $\mathbb{R}^3$ is in W–for example, the vector $\langle 5, -5, 1 \rangle$ graphed in blue below:



Since W is clearly a subset of the vector space $\mathbb{R}^3$, we would like to know if W, equipped with the usual addition and scalar multiplication, is a *subspace* of $\mathbb{R}^3$. According to the theorem, we simply need to check that:

1. $\mathbf{0} \in W$
2. If u and v are vectors in W, then $u + v$ is a vector in W.
3. If λ is a scalar and u is a vector in W, then λu is a vector in W.

Let's check:

1. $\boldsymbol{0} \in W$: Clearly

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has the right form and is in W.

2. *If u and v are vectors in W, then $u+v$ is a vector in W*: Let

$$u = \langle x_1, x_1 + z_1, z_1\rangle \text{ and } v = \langle x_2, x_2 + z_2, z_2\rangle.$$

We need to check and see if the vector $u+v$ is also in W, so let's add the vectors:

$$\begin{aligned} u+v &= \langle x_1, x_1 + z_1, z_1\rangle + \langle x_2, x_2 + z_2, z_2\rangle \\ &= \langle x_1 + x_2, x_1 + z_1 + x_2 + z_2, z_1 + z_2\rangle \\ &= \langle x_1 + x_2, (x_1 + x_2) + (z_1 + z_2), z_1 + z_2\rangle. \end{aligned}$$

Notice that the second coordinate of the vector $u+v$ is the sum of the first and third coordinates; so $u+v$ obeys the rule for W, and is indeed a vector in W.

3. *If λ is a scalar and u is a vector in W, then λu is a vector in W*: With

$$u = \langle x_1, x_1 + z_1, z_1\rangle,$$

we need to calculate λu:

$$\begin{aligned} \lambda u &= \lambda\langle x_1, x_1 + z_1, z_1\rangle \\ &= \langle \lambda x_1, \lambda(x_1 + z_1), \lambda z_1\rangle \\ &= \langle \lambda x_1, \lambda x_1 + kz_1, \lambda z_1\rangle. \end{aligned}$$

Again, we see that the second coordinate of the vector λu is the sum of the first and third coordinates, and is thus a vector in W.

Since W passes all of the tests of the theorem, it is a subspace of $\mathbb{R}^3$.

Example 2

In the previous section, we saw that the set $\mathbb{R}(-\infty,\infty)$, whose vectors are real-valued functions defined on $(-\infty,\infty)$, is a vector space. I claim that the subset $C(-\infty,\infty)$ of *continuous* real-valued functions is a subspace of $\mathbb{R}(-\infty,\infty)$.

Of course, this claim is quite easy to check: we know from calculus that the sum $f(x)+g(x)$ of two continuous functions is also a continuous function, as is the product $\lambda f(x)$ of a scalar and a continuous function. In addition, the function 0 is continuous; thus $C(-\infty,\infty)$ is a subspace of $\mathbb{R}(-\infty,\infty)$.

Example 3

Recall that the transpose of a 2×2 matrix

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ is } X^\top = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}.$$

Let X be a 2×2 matrix that equals its transpose, i.e. $X = X^\top$; recall that such matrices are referred to as *symmetric*. In terms of a formula, we see that

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} = X^\top,$$

so that we must have $x_{12} = x_{21}$.

For example, given the matrix

$$X = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix},$$

we see that

$$X^\top = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} = X,$$

so that X is symmetric.

Let $S_2(\mathbb{F})$ be the set of all 2×2 matrices with entries in $\mathbb{F}$ that equal their transposes, i.e. all of the matrices of the form

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}.$$

Clearly this is a subset of the vector space $\mathcal{M}_2(\mathbb{F})$ of all 2×2 matrices, and I claim that S_2 is actually a *subspace* of $\mathcal{M}_2(\mathbb{F})$.

To verify that S_2 is a subspace, we once again check the three conditions of the theorem:

1. $\boldsymbol{0} \in S_2$: Clearly $\mathbf{0}$ is symmetric:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2. *Closure under addition*: Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}.$$

We need to be certain that the vector $X + Y$ is also in S_2; in other words, that $(X + Y)^\top = X + Y$. Fortunately, there is a quick way to do this: we learned earlier in the course that, for any pair of matrices whose sizes are amenable for addition,

$$(X + Y)^\top = X^\top + Y^\top.$$

Now since our X and Y are in S_2, so that

$$X = X^\top \text{ and } Y = Y^\top,$$

we see that

$$\begin{aligned}(X+Y)^\top &= X^\top + Y^\top \\ &= X+Y.\end{aligned}$$

So we see that

$$(X+Y)^\top = X+Y,$$

so that $X+Y$ is a vector in S_2, as required.

3. *Closure under scalar multiplication*: Again, the calculation here is quite easy to make. We wish to be certain that λX is a vector in S_2, i.e. that

$$(\lambda X)^\top = \lambda X.$$

Of course, we know that scalars are unaffected by taking transposes, so that

$$(\lambda X)^\top = \lambda X^\top.$$

Together with the fact that X itself is in S_2, $X = X^\top$, we have

$$\begin{aligned}(\lambda X)^\top &= \lambda X^\top \\ &= \lambda X,\end{aligned}$$

so that λX is a vector in S_2.

Since the elements of S_2 satisfy the requirements of the theorem, S_2 is a subspace of $\mathcal{M}_2$, and a vector space in its own right.

Example 4

Recall that the *trace* of a square matrix is the sum of its diagonal entries.

Let $\mathfrak{sl}(2,\mathbb{R})$ be the set of all matrices in $\mathcal{M}_2(\mathbb{R})$ with trace 0. For example,

$$\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix}, \quad \begin{pmatrix}5 & 7\\ -3 & -5\end{pmatrix}, \text{ and } \begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}$$

are all vectors in $\mathfrak{sl}(2,\mathbb{R})$.

I claim that $\mathfrak{sl}(2,\mathbb{R})$ is a subspace of $\mathcal{M}_2(\mathbb{R})$. Again, we check the conditions of the theorem:

1. $\mathbf{0} \in \mathfrak{sl}(2,\mathbb{R})$: Clearly $\operatorname{tr}\mathbf{0} = 0$.

2. *Closure under addition*: Since X, $Y \in \mathfrak{sl}(2,\mathbb{R})$, say

$$X = \begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}, \quad Y = \begin{pmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix},$$

we know that

$$x_{11} + x_{22} = 0 \text{ and } y_{11} + y_{22} = 0.$$

Since the diagonal entries of $X + Y$ are

$$x_{11} + y_{11} \text{ and } x_{22} + y_{22},$$

we see that

$$\begin{aligned} \operatorname{tr}(X+Y) &= x_{11} + y_{11} + x_{22} + y_{22} \\ &= x_{11} + x_{22} + y_{11} + y_{22} \\ &= 0, \end{aligned}$$

so that $X + Y \in \mathfrak{sl}(2, \mathbb{R})$.

3. *Closure under scalar multiplication*: Similarly,

$$\begin{aligned} \operatorname{tr}(kX) &= kx_{11} + kx_{22} \\ &= k(x_{11} + x_{22}) \\ &= k \cdot 0 \\ &= 0, \end{aligned}$$

so $kX \in \mathfrak{sl}(2, \mathbb{R})$ as required.

Remark. The last example actually illustrates some important properties of the trace function that we will not discuss in detail–namely, that $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, and that $\operatorname{tr}(kA) = k \cdot \operatorname{tr} A$.

Examples of Subsets that are Not Subspaces

Example 1

Let W be the set of all vectors in $\mathbb{R}^2$ of the form

$$\langle x, 1 \rangle.$$

W is clearly a *subset* of $\mathbb{R}^2$ (in fact, it is the horizontal line at y=1); however, it is not not a subspace of $\mathbb{R}^2$, since the sum

$$\langle x, 1 \rangle + \langle y, 1 \rangle = \langle x + y, 2 \rangle$$

of any pair of vectors in W will have a 2 as its second coordinate, thus will not be an element of W.

Example 2

Let W be the set of all 3×3 matrices with determinant 1. Even though this set is a subset of $\mathcal{M}_3(\mathbb{C})$, it is definitely not a subspace, as it fails the second condition of the theorem: *If λ is a scalar and u is a vector in W, then λu is a vector in W.*

As an example, consider the determinant 1 matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W.$$

Clearly iI *is not* a determinant 1 matrix–indeed, you should check that $\det(iI) = -i$. Thus W fails the third condition of the theorem.

Sums of Subspaces

We can use subspaces of a vector space to build *more* subspaces; one way to do so is by creating a *sum* of subspaces.

Definition 1.36. Let U_1, U_2, ..., U_n be subspaces of a vector space V. The *sum* of U_1, U_2, ..., U_n is the set of all possible sums of elements of the subspaces U_1, U_2, ..., U_n; we write

$$U_1 + U_2 + \ldots + U_n = \{u_1 + u_2 + \ldots + u_n \mid u_1 \in U_1,\ u_2 \in U_2,\ \ldots,\ u_n \in U_n\}.$$

Remark. As we begin to think about sums of subspaces, it is important to note exactly what types of objects live in the sum: note that $U_1 + U_2 + \ldots + U_n$ is a *set of vectors*; every vector in this set is made up of sums of vectors from the component subspaces. So at the very least, we know that the set $U_1 + U_2 + \ldots + U_n$ is a subset of some of the vectors of the ambient vector space V.

Example. Let $V = \mathbb{R}^3$, the vector space of all triplets of real numbers, which we think of geometrically as three-dimensional space, with the usual operations of vector addition and scalar multiplication.

It is easy to check that

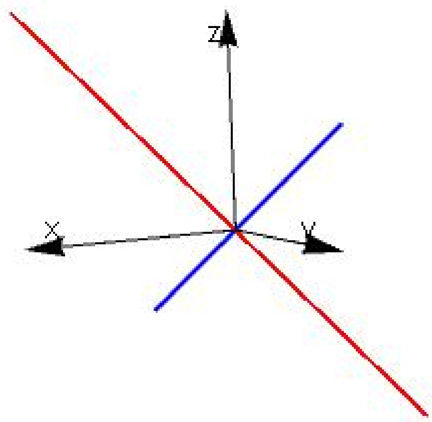
$$U_1 = \left\{ \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} \middle| r \in \mathbb{R} \right\}$$

and

$$U_2 = \left\{ \begin{pmatrix} 0 \\ t \\ \frac{t}{2} \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

are both subspaces of V (indeed, you should check using the theorem).

We plot the two subspaces below, U_1 in red and U_2 in blue:



We are going to build the sum $U_1 + U_2$ of the two vector spaces; before investigating it in general, let's think about a few examples of vectors in the set $U_1 + U_2$.

First of all, we know that $\mathbf{0} \in U_2$, so for any vector $u \in U_1$,

$$u + \mathbf{0} = u \in U_1 + U_2.$$

In other words,

$$U_1 \subset U_1 + U_2.$$

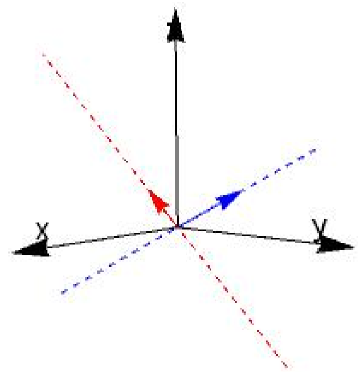
Similarly,

$$U_2 \subset U_1 + U_2.$$

However, the set $U_1 + U_2$ will contain many vectors that are in neither U_1 nor U_2. For example, we know that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in U_1 \text{ and } \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \in U_2;$$

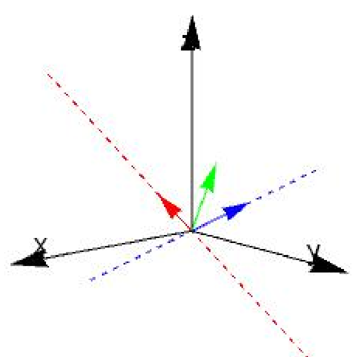
these two vectors are graphed below in red and blue respectively:



Their sum

$$\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

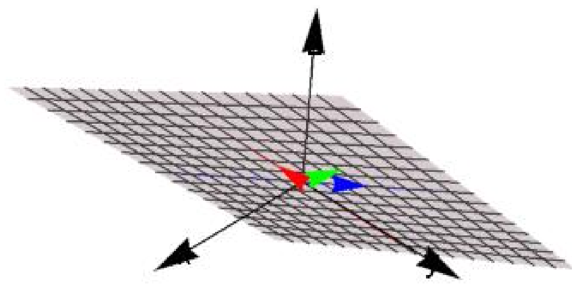
must also be a vector in $U_1 + U_2$; the vector is graphed below in green:



At this point, you may have already guessed the punch line: $U_1 + U_2$ is the plane in $\mathbb{R}^3$ passing through the two lines, since $U_1 + U_2$ consists of all vectors of the form

$$\begin{pmatrix}r\\t\\r+\frac{t}{2}\end{pmatrix},$$

a parametric equation for a plane. The plane $U_1 + U_2$ is graphed below:



You may have made another important observation about this example: $U_1 + U_2$ is not merely a *subset* of vectors from $\mathbb{R}^3$–it is a *subspace* of $\mathbb{R}^3$.

It turns out that the observation we made in the previous example is true in general, as indicated by the next theorem:

Theorem 1.39. If $U_1, U_2, \ldots, U_n$ are subspaces of a vector space V, then their sum $U_1+U_2+\ldots+U_n$ is also a subspace of V, and is the smallest subspace of V containing all of $U_1, U_2, \ldots, U_n$.

Proof. *Exercise.*

Example. Let U_1 be the subspace of all diagonal matrices in $\mathcal{M}_2(\mathbb{F})$, and U_2 be the subspace of all upper triangular matrices in $\mathcal{M}_2(\mathbb{F})$. Describe the vector space $U_1 + U_2$.

Let's begin by describing the subspaces U_1 and U_2 in detail:

$$U_1 = \left\{ \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \middle| d_{11}, d_{22} \in \mathbb{C} \right\},$$

and

$$U_2 = \left\{ \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \middle| u_{11}, u_{12}, u_{22} \in \mathbb{C} \right\}.$$

Thus the vector space $U_1 + U_2$ consists of all matrices of the form

$$\begin{pmatrix} u_{11} + d_{11} & u_{12} \\ 0 & u_{22} + d_{22} \end{pmatrix};$$

but u_{11}, d_{11}, etc., were arbitrary in the first place, so $U_1 + U_2$ merely consists of all upper triangular 2×2 matrices. That is,

$$U_1 + U_2 = U_2.$$

The last example is rather interesting: the sum of a pair of subspaces turned out to be the larger subspace. This happened because U_1 itself was actually already a subspace of U_2, so we didn't create any "new" vectors by adding U_1 to U_2.

Looking ahead, we will eventually want to *avoid* such sums, as they introduce unpleasant ambiguities. For example, the vector

$$\begin{pmatrix} i & 1+i \\ 0 & 2-i \end{pmatrix}$$

can be written in multiple ways as a sum of vectors in U_1 and U_2: we could write

$$\begin{pmatrix} i & 1+i \\ 0 & 2-i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 2-i \end{pmatrix} + \begin{pmatrix} 0 & 1+i \\ 0 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} i & 1+i \\ 0 & 2-i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} 2i & 1+i \\ 0 & 2 \end{pmatrix};$$

in both examples, the first summand is in U_1 and the second is in U_2. There is no unique way to decompose elements of the sum using the summands.

With this idea in mind, we define a special type of sum, the *direct sum*:

Definition 1.40. Let U_1, U_2, ..., U_n be subspaces of a vector space V.

- The sum $U_1 + U_2 + \ldots + U_n$ is called a *direct sum* if each vector u in the sum can be written uniquely as

  $$u = u_1 + u_2 + \ldots + u_n,$$

  $u_1 \in U_1$, ..., $u_n \in U_n$.

- If the sum $U_1 + U_2 + \ldots + U_n$ *is* a direct sum, we write

  $$U_1 \oplus U_2 \oplus \ldots \oplus U_n,$$

  where $\oplus$ indicates that the sum is direct.

Example. We have actually already seen an example of a direct sum; let $V = \mathbb{R}^3$,

$$U_1 = \left\{ \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} \middle| r \in \mathbb{R} \right\},$$

and

$$U_2 = \left\{ \begin{pmatrix} 0 \\ t \\ \frac{t}{2} \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

Earlier, we saw that the vector space $U_1 + U_2$ can be described by

$$U_1 + U_2 = \left\{ \begin{pmatrix} r \\ t \\ r + \frac{t}{2} \end{pmatrix} \middle| r, s \in \mathbb{R} \right\}.$$

Every vector in $U_1 + U_2$ thus has form

$$\begin{pmatrix} r \\ t \\ r + \frac{t}{2} \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ \frac{t}{2} \end{pmatrix};$$

clearly there is *only one way* to choose r and t. Thus the decomposition is unique; we may now write

$$U_1 \oplus U_2$$

to indicate that the sum is direct.

It would be helpful to have further criteria for testing whether or not a sum is direct; the next theorem provides this to us:

Theorem 1.44. Let U_1, U_2, ..., U_n be subspaces of a vector space V. Then the sum $U_1 + U_2 + \ldots + U_n$ is direct if and only if the only way to write $\mathbf{0}$ as a sum of elements u_1, ..., u_n of U_1, ..., U_n respectively, is to choose $u_i = \mathbf{0}$ for all i.

Proof. $\Longrightarrow$: If the sum is direct, then by definition the decomposition

$$\mathbf{0} = u_1 + u_2 + \ldots + u_n,$$

where $u_i \in U_i$, is unique. Since $\mathbf{0} \in U_i$ for all i, we must have $u_i = \mathbf{0}$, all i.

$\Longleftarrow$: On the other hand, let $u \in U_1 + \ldots + U_n$, and suppose that

$$u = u_1 + \ldots + u_n \text{ and } u = u_1' + \ldots + u_n'$$

are two decompositions of u, with $u_1, u_1' \in U_1$, ..., $u_n, u_n' \in U_n$. Consider the quantity

$$\begin{aligned} \mathbf{0} &= u - u \\ &= u_1 + \ldots + u_n - (u_1' + \ldots + u_n') \\ &= (u_1 - u_1') + \ldots + (u_n - u_n)'. \end{aligned}$$

Clearly $u_i - u_i' \in U_i$; thus the equation

$$\mathbf{0} = (u_1 - u_1') + \ldots + (u_n - u_n)'$$

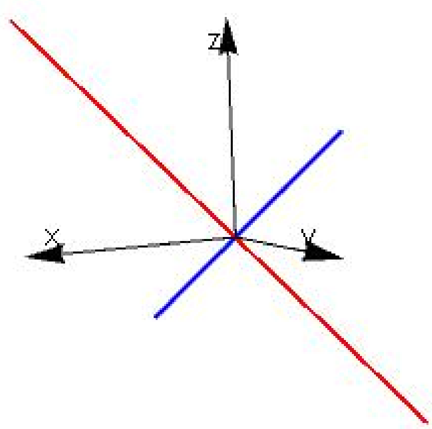
is a decomposition of $\mathbf{0}$ into a sum of elements of $U_1, \ldots, U_n$. By assumption, the *only* way to write $\mathbf{0}$ as a sum of elements of $U_1, \ldots, U_n$ is to choose $u_i - u_i' = 0$. Thus $u_i = u_i'$, and the decomposition of u is unique.

In certain cases, there is an even simpler way to determine whether or not a sum is direct.

Theorem 1.45. If U and W are subspaces of V, then $U + V$ is a direct sum if and only if $U \cap V = \{\mathbf{0}\}$.

Proof. Exercise.

We can illustrate the theorem graphically with the example we have inspected multiple times in this section:



The red and blue lines above form subspaces of $\mathbb{R}^3$, and since their intersection is identically $\{\mathbf{0}\}$, their sum is direct (as we verified earlier).

It is tempting to think that the theorem could be generalized–namely, we might hope that $U_1 + \ldots + U_n$ is a direct sum if and only if $U_i \cap U_j = \{\mathbf{0}\}$ whenever $i \neq j$. Unfortunately, this condition is not enough to guarantee that the sum is direct, as indicated by the following example.

Example. The vector space $\mathcal{U}_2(\mathbb{R})$ of all 2×2 upper triangular matrices with real entries has subspaces

$$U_1 = \left\{ \begin{pmatrix} x & x \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{R} \right\},$$

$$U_2 = \left\{ \begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix} \middle| y \in \mathbb{R} \right\},$$

and

$$U_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

Now the intersection of any pair of of subspaces is $\{\mathbf{0}\}$; for example, if $u \in U_1 \cap U_2$, say

$$u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix},$$

then $u_{22} = 0$ since $u \in U_2$; then since $u_{22} = 0$, we must have $u_{11} = u_{12} = u_{22} = 0$ since $u \in U_1$.

However, the sum $U_1 + U_2 + U_3$ is not direct; in particular, it is easy to decompose $\mathbf{0}$ as a sum of nonzero elements of U_1, U_2, and U_3, say as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$