Vector Spaces

In your Calculus courses, you discussed many of the special properties that are shared by the different versions of Euclidean space, such as the plane ($\mathbb{R}^2$), three dimensional space ($\mathbb{R}^3$), etc. We saw that, across the different varieties of Euclidean space, vectors, the objects which make up the spaces, behave virtually the same way. For example, we know that vector addition is commutative, that is

$$u + v = v + u,$$

regardless of whether u and v are vectors in $\mathbb{R}^2$, $\mathbb{R}^5$, or even $\mathbb{R}^{100}$.

Euclidean space is our motivation for the definition of a *vector space*. In this section, we will introduce the idea of vector spaces and see that such spaces share a great deal of algebraic and geometric structure with Euclidean space.

Definitions 1.18-1.19. A *vector space V over a field* $\mathbb{F}$ is a set V of objects call *vectors*, along with two operations defined on V:

1. **Closure of V under addition**: the operation "+" assigns a vector $u + v$ in V to every pair u, v of vectors in V.

2. **Closure of V under scalar multiplication**: the operation of scalar multiplication assigns a vector λu to every pair λ, u, where λ is an element of $\mathbb{F}$ and u is a vector in V.

The operations of vector addition and scalar multiplication must satisfy the following rules:

3. **Commutativity:** $u + v = v + u \ \forall \ u, v \in V$

4. **Associativity:** $(u + v) + w = u + (v + w) \ \forall \ u, v, w \in V$

5. **Additive Identity:** There is an element $\mathbf{0}$ in V, called the *zero vector*, so that

$$u + \mathbf{0} = \mathbf{0} + u = u$$

   for every $u \in V$.

6. **Additive Inverse:** For each elemnt u in V, there is another element w in V, called an *additive inverse* of u, so that $u + w = \mathbf{0}$.

7. **Multiplicative Identity:** The number $1 \in \mathbb{F}$ has the property that $1u = u$ for all $u \in V$.

8. **Distribution of Scalar Multiplication over Vector Addition:** $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in V$, all $\alpha \in \mathbb{F}$.

9. **Distribution of Scalar Multiplication over Scalar Addition:** $(\alpha + \gamma)u = \alpha u + \gamma u$ for all $u \in V$, all $\alpha, \gamma \in \mathbb{F}$.

Remark 1. In this class, our scalars will always be elements of $\mathbb{C}$ or of $\mathbb{R}$, that is either complex or real numbers. When we need to distinguish between the two, we will refer to a particular vector space as a *vector space over* $\mathbb{C}$ or as a *vector space over* $\mathbb{R}$. We will write $\mathbb{F}$ to indicate either $\mathbb{C}$ or $\mathbb{R}$ when it is not necessary to draw a distinction.

Remark 2. We will not have time in this course to discuss the precise definition of a field (although you would certainly encounter the definition in a second course in abstract algebra). For our purposes, it is enough to know that $\mathbb{R}$ and $\mathbb{C}$ are fields; both have an addition operation as well as a multiplication operation, and these operations obey a number of useful rules.

Remark 3. You should be careful to note that the definition above of a vector space *does not* specify the way in which "addition" works, and when we discuss a particular space, we will need to note exactly what we mean by "+".

Examples of Vector Spaces

Throughout the rest of this class, you will need to become comfortable with determining whether or not a set forms a vector space under the given operations of addition and scalar multiplication. To be technically correct, you must check that the set satisfies *all* of the conditions from the definition before you can be certain that the set is indeed a vector space.

In practice, however, we will often check just a few of the properties, as checking all of them will take an abundance of time that could be better spent on other topics.

Example 1: $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the set of all n-tuples of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where each x_i is a real number. Define addition using usual vector addition, that is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix},$$

and scalar multiplication by

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

It is quite easy to check that $\mathbb{R}^n$ is a field over $\mathbb{R}$. Let's briefly discuss the required properties:

1. **Closure under addition**: $\mathbb{R}^n$ is clearly closed under addition: the sum of two vectors is another vector in $\mathbb{R}^n$.

2. **Closure under scalar multiplication**: again, $\mathbb{R}^n$ is clearly closed under the operation: the product of a real number and a vector is another vector in $\mathbb{R}^n$.

3. **Commutativity:** Since addition of real numbers is commutative, we see that $\mathbb{R}^n$ has the commutative property as well:

$$\begin{aligned} x+y &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} \\ &= \begin{pmatrix} y_1+x_1 \\ y_2+x_2 \\ \vdots \\ y_n+x_n \end{pmatrix} \\ &= \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= y+x. \end{aligned}$$

4. **Associativity:** The reasoning for associativity is similar to that for commutativity–real number addition is associative.

5. **Additive Identity:** The $\mathbf{0}$ vector in $\mathbb{R}^n$ is

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

   it is easy to check that $x + \mathbf{0} = x$ for every vector $x \in \mathbb{R}^n$.

6. **Additive Inverse:** Given

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$$

I claim that

$$-x := \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$$

is an additive inverse. We can check the claim by showing that $x + (-x) = \mathbf{0}$:

$$\begin{aligned} x + (-x) &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 - x_1 \\ x_2 - x_2 \\ \vdots \\ x_1 - x_n \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

7. **Multiplicative Identity:** We need to check that $1x = x$ for every vector $x \in \mathbb{R}^n$ (where 1 indicates the *number* 1 in $\mathbb{R}$):

$$\begin{aligned} 1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \\ \vdots \\ 1 \cdot x_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x. \end{aligned}$$

8. **Distribution of Scalar Multiplication over Vector Addition:** Easy to check.

9. **Distribution of Scalar Multiplication over Scalar Addition:** Also easy to check.

Since $\mathbb{R}^n$ satisfies all of the properties from the definition, it is a vector space over $\mathbb{R}$.

Remark 4. Having seen our first example of a vector space, it is a good time to point out that we generally need to specify the field over which we are working. In particular, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, but it *is not* a vector space over the field $\mathbb{C}$.

To understand why not, think for a moment about scalar multiplication; given a vector x in $\mathbb{R}^n$, the scalar product of $i \in C$ with x is a vector ix whose entries are *complex numbers*; that is, $ix \notin \mathbb{R}^n$.

Example 2: $\mathbb{C}^n$

The set $\mathbb{C}^n$ of all n-tuples of complex numbers, with the standard vector addition and scalar multiplication as defined above, is also a field. It is interesting to note that $\mathbb{C}^n$ is a field over $\mathbb{R}$, as well as over $\mathbb{C}$.

Example 3: $\mathcal{M}_{mn}(\mathbb{F})$

The set $\mathcal{M}_{mn}$ of all $m \times n$ matrices with entries from $\mathbb{F}$ is a vector space over $\mathbb{F}$, with the usual operations of matrix addition and scalar multiplication. We write $\mathcal{M}_n(\mathbb{F})$ to refer to the vector space of all $n \times n$ matrices.

Example 4: The set of all polynomials of at most degree n with coefficients in $\mathbb{F}$

Recall that an nth degree polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where each of the constants a_i is a number in $\mathbb{F}$, and $a_n \neq 0$.

The set $\mathcal{P}_n(\mathbb{F})$ of all polynomials of at most degree n is a vector space over $\mathbb{F}$, with the usual polynomial addition and scalar multiplication as its operations. So objects like

$$p_1(x) = ix^5 - 12x,\ p_2(x) = 4x^2 - ix + 1,\ \text{and}\ p_3(x) = 0$$

are all vectors in, say, $\mathcal{P}_5(\mathbb{C})$, and we can easily add them:

$$p_1(x) + p_2(x) = ix^5 - 12x + 4x^2 - ix + 1 = ix^5 + 4x^2 - (12 + i)x + 1.$$

Multiplication of $p_1(x)$ by the scalar $-i$ looks like

$$-ip_1(x) = -i(ix^5 - 12x) = x^5 + 12ix.$$

Example 5: The set of all real-valued functions

Let $\mathbb{R}(-\infty, \infty)$ denote the set of all functions $f(x)$ defined on the interval $(-\infty, \infty)$ whose outputs are real numbers.

Given a pair f and g of functions in $\mathbb{R}(-\infty, \infty)$, we define the addition operation by setting $f + g$ equal to the function that evaluates to $f(x) + g(x)$, that is

$$(f + g)(x) = f(x) + g(x).$$

The scalar multiple kf is the function defined by its evaluation at x by the rule

$$(kf)(x) = kf(x).$$

$\mathbb{R}(-\infty, \infty)$ is a vector space over $\mathbb{R}$.

Example 6: The set of 2×2 upper triangular matrices

Let $\mathcal{U}_2$ be the set of all upper triangular 2×2 matrices with entries in $\mathbb{F}$, i.e. the set of all matrices of the form

$$\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}.$$

It is not too hard to verify that this set, together with the usual operations of matrix addition and scalar multiplication, is a vector space over $\mathbb{F}$.

Let's quickly verify axioms 1 and 2:

1. **Closure under addition**: if u and v are in $\mathcal{U}_2$, they must be 2×2 upper triangular matrices, of the form

$$u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \text{ and } v = \begin{pmatrix} v_{11} & v_{12} \\ 0 & v_{22} \end{pmatrix}.$$

   We should check to see if their sum is also a 2×2 upper triangular matrix:

$$\begin{aligned} u + v &= \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ 0 & v_{22} \end{pmatrix} \\ &= \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ 0 & u_{22} + v_{22} \end{pmatrix}, \end{aligned}$$

   which is clearly an upper triangular 2×2 matrix; thus $\mathcal{U}_2$ is closed under addition.

2. **Closure under scalar multiplication**: Given a matrix

$$u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

from $\mathcal{U}_2$ and any scalar k, we need to make sure that ku is also in $\mathcal{U}_2$:

$$\begin{aligned} \lambda u &= \lambda \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \\ &= \begin{pmatrix} \lambda u_{11} & \lambda u_{12} \\ 0 \cdot \lambda & \lambda u_{22} \end{pmatrix} \\ &= \begin{pmatrix} \lambda u_{11} & \lambda u_{12} \\ 0 & k u_{22} \end{pmatrix}, \end{aligned}$$

another upper triangular 2×2 matrix with entries in $\mathbb{F}$. Thus $\mathcal{U}_2$ is closed under scalar multiplication.

In order to be certain that $\mathcal{U}_2$ is a vector space, we should check the remaining vector space properties; however, the process is tedious and we will skip it in the notes. $\mathcal{U}_2$ does indeed pass all of them, thus is a vector space.

Examples of Sets That are Not Vector Spaces

Example 1: Integers with the usual addition and scalar multiplication

The set $\mathbb{Z}$ of all real integers, equipped with the normal notions of addition and scalar multiplication, is *not* a vector space over the real numbers. The main problem here is axiom 2, *closure under scalar multiplication.*

For example, the number 4 is an element of $\mathbb{Z}$, but not all of its scalar multiples are; choosing scalar $\lambda = 1/5$, we see that

$$\lambda \cdot 4 = \frac{4}{5},$$

which is not an element of the set $\mathbb{Z}$ of all real integers.

Example 2: The set of all degree 5 polynomials

Earlier in this section, we saw that of the set $\mathcal{P}_n(\mathbb{F})$ of all polynomials of degree *at most* n is a vector space over $\mathbb{F}$; we considered $\mathcal{P}_5$ in particular. I claim that the set of all polynomials with degree *exactly* 5 with the usual polynomial addition and scalar multiplication *is not* a vector space.

To be clear, this set consists of all polynomials of the form

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

where $a_5 \neq 0$ and each of the other a_i can be any number in $\mathbb{F}$. So elements of this set look like

$$2.5x^5 - 3x,\ x^5 + x^4 + 1,\ \text{and}\ -x^5;$$

but the following polynomials are *not* in the set, since they do not include a degree 5 term:

$$x^3 - 1,\ 5x^4,\ \text{and}\ 0.$$

You may have already guessed why this set is not a vector space–it fails to be closed under addition.

To see why this is the case, consider the elements $p = x^5 - x^4$ and $r = -x^5$. Both are degree 5 polynomials, thus in our set, but their sum

$$p + r = x^5 - x^4 - x^5 = -x^4$$

is most definitely *not* a degree 5 polynomial, thus not in the set. Thus the set of all degree 5 polynomials is not a vector space with the given operations.

Uniqueness of Identity and Inverses

We should take a moment to investigate the ideas of *identity* and *additive inverse* a bit more closely. Specifically, the definition above for a vector space does not decree that there is a single identity element; indeed, there could potentially be several vectors in a given vector space that act like the identity.

Fortunately, it turns out that this is not the case: the additive identity is unique, as indicated by the following theorem:

Theorem 1.25. The additive identity $\mathbf{0}$ in V is unique.

Proof. Suppose that there are two elements $\mathbf{0}$ and $\mathbf{0'}$ of V which act as additive identity, that is

$$v + \mathbf{0} = v \text{ and } v + \mathbf{0'} = v\ \forall\ v \in V.$$

In particular,

$$\mathbf{0} + \mathbf{0'} = \mathbf{0}$$

since $\mathbf{0'}$ is an additive identity; but

$$\mathbf{0} + \mathbf{0'} = \mathbf{0'}$$

using the same reasoning. Thus

$$\mathbf{0} = \mathbf{0} + \mathbf{0'} = \mathbf{0'},$$

so that the additive identity is unique.

Along similar lines of reasoning, we might be concerned about the uniqueness of additive inverses. That is, is it possible that two different elements v and w of vector space V can “cancel” the same vector u:

$$u + v = \mathbf{0} \text{ and } u + w = \mathbf{0}?$$

Again, the vector space axioms prevent this ambiguity from becoming an issue, as indicated by the following theorem:

Theorem 1.26. Every element u of a vector space V has a unique additive inverse.

Proof. Suppose that v and w are both additive inverses of u, that is

$$u + v = \mathbf{0} \text{ and } u + w = \mathbf{0}.$$

Then

$$\begin{aligned} u + v &= u + w \\ v + (u + v) &= v + (u + w) \\ (v + u) + v &= (v + u) + w \\ \mathbf{0} + v &= \mathbf{0} + w \\ v &= w. \end{aligned}$$

Thus the additive inverse of an element u of V is unique.

Since additive inverses are unique, we will use special notation to indicate the additive inverse of a particular element:

$$u \text{ has additive inverse } -u.$$

We may now use notation such as

$$v - u$$

to indicate

$$v - u = v + (-u).$$

Properties of the Vector $\mathbf{0}$ and Scalars 0 and 1

Given our knowledge of number arithmetic and the properties of $\mathbb{R}$ and $\mathbb{C}$, we tend to take several properties of the $\mathbf{0}$ vector and scalars 0 and 1 for granted. However, since we are working with new, abstract mathematical objects–vector spaces–we should confirm that these objects actually do behave the way that we expect them to.

We know that the number 0 is the “annihilator”: it zeros out every other number under the multiplication operation. It turns out that 0 behaves in a similar fashion when combined with vectors under scalar multiplication:

Theorem 1.29. Given any vector $u \in V$, $0u = \mathbf{0}$.

Proof. Using properties of number arithmetic, we know that

$$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u. \end{aligned}$$

Now the vector $0u$ has a unique additive inverse, $-0u$; adding this vector to both sides of the equation, we see that

$$\begin{aligned} 0u - 0u &= (0u + 0u) - 0u \\ &= 0u + (0u - 0u) \\ &= 0u + \mathbf{0} \\ &= 0u. \end{aligned}$$

We see that

$$\mathbf{0} = 0u - 0u = 0u,$$

proving the theorem.

In keeping with the ideas above, it turns out that the *vector* $\mathbf{0}$ has a similar property to the *number* 0 under scalar multiplication:

Theorem 1.30. Given any scalar λ in $\mathbb{F}$,

$$\lambda\mathbf{0} = \mathbf{0}.$$

Proof. Given λ in $\mathbb{F}$ and $u \in V$, we know that

$$\begin{aligned} \lambda u &= \lambda(u + \mathbf{0}) \\ &= \lambda u + \lambda\mathbf{0}; \end{aligned}$$

but using the unique additive inverse $-\lambda u$ of λu, we see that

$$\lambda u = \lambda u + \lambda\mathbf{0}$$

implies that

$$\begin{aligned} \lambda u - \lambda u &= \lambda u + \lambda\mathbf{0} - \lambda u \\ \mathbf{0} &= \lambda u - \lambda u + \lambda\mathbf{0} \\ \mathbf{0} &= \mathbf{0} + \lambda\mathbf{0} \\ \mathbf{0} &= \lambda\mathbf{0}, \end{aligned}$$

proving the claim.

Finally, we make concrete the relationship between the additive inverse $-u$ of vector u, and the element $(-1)u$:

Theorem 1.31. Given any vector $u \in V$, $(-1)u = -u$.

Proof. Additive inverses are unique, so we merely need to prove that $(-1)u$ *is* the additive inverse of u. Clearly

$$\begin{aligned} u + (-1)u &= (1-1)u \\ &= 0u \\ &= \mathbf{0}. \end{aligned}$$

Thus $(-1)u$ is the additive inverse of u, that is

$$(-1)u = -u.$$

Again, we note that Theorem 1.31 expands the role of the number -1 to apply to vector spaces: in the world of numbers, of course, we know that $(-1)a$ is the additive inverse of a. The same reasoning applies to vectors in a vector space.