Invertibility and Properties of Determinants

In a previous section, we saw that the trace function, which calculates the sum of the diagonal entries of a square matrix, interacts nicely with the operations of matrix addition, scalar multiplication, and matrix multiplication. We would like to know how the determinant function interacts with these operations as well.

In other words, if we know $\det A$ and $\det B$, can we use this information to find quantities such as

$$\det(kA),\ \det(A+B),\ \text{and}\ \det AB?$$

It turns out that the answers to the first and third questions are quite easy to find, whereas (perhaps surprisingly), the answer to the second question is actually quite difficult, and is the topic of much current research in linear algebra (including some of my own).

In this section, we will also answer the question we raised in section 6:

How can we tell whether or not a particular square matrix A has an inverse?

Determinants and Scalar Multiplication

Let's think about the first question raised above: If we know the determinant of matrix A, can we use this information to calculate the determinant of the matrix kA, where k is a constant?

Fortunately, there is an easy answer to this question:

Theorem. If A is an $n \times n$ matrix, and k is any constant, then

$$\det kA = k^n \det A.$$

We cannot yet prove this theorem; we will investigate it in more detail once we have the right tools.

Determinants and Matrix Addition

Our next question, about the relationship between $\det(A+B)$, $\det A$, and $\det B$, does not have a satisfactory answer, as indicated by the following example:

Example

Let

$$A = \begin{pmatrix} 5 & -6 \\ 0 & -12 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 0 \\ 1 & 9 \end{pmatrix}.$$

Compare $\det A$, $\det B$, and $\det(A+B)$.

The determinants of A and B are quite simple to calculate, given that they are triangular matrices: clearly $\det A = -60$, and $\det B = -27$. Now

$$A + B = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix};$$

since $A+B$ is not a triangular matrix, we'll need to revert to the formula for determinants of 2×2 matrices to calculate $\det(A+B)$:

$$\begin{aligned} \det(A+B) &= \det \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} \\ &= 2 \cdot (-3) - (-6) \cdot 1 \\ &= -6 + 6 \\ &= 0. \end{aligned}$$

Gathering our data, we see that

$$\det A = -60, \ \det B = -27, \text{ and } \det(A+B) = 0.$$

Unfortunately, it appears that there is very little connection between $\det A$, $\det B$, and $\det(A+B)$.

Key Point. In general,

$$\det A + \det B \neq \det(A+B),$$

and you should be *extremely* careful not to assume anything about the determinant of a sum.

Nerdy Sidenote

One large vein of current research in linear algebra deals with this question of how $\det A$ and $\det B$ relate to $\det(A+B)$. One way to handle the question is this: instead of trying to find the value for $\det(A+B)$, find a region in the complex plane that we can be certain *contains* $\det(A+B)$. It turns out that this is a rich question with many interesting and surprising answers. Some of my own research relates to this question.

Determinants and Matrix Multiplication

Perhaps surprisingly, determinants of products are quite easy to compute:

Theorem. If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B).$$

In other words, the determinant of a product of two matrices is just the product of the determinants. We are not yet ready for a proof of the theorem, but will return to it when we have the proper tools.

Example

Compute $\det AB$, given

$$A = \begin{pmatrix} 5 & -6 \\ 0 & -12 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 0 \\ 1 & 9 \end{pmatrix}$$

from the previous example.

According to the theorem above, there are two ways to handle this problem:

1. Multiply A by B, then calculate the determinant of the product.
2. Find determinants of A and B separately, then multiply them together to get the determinant of AB.

Of course, it is easy to see that

$$\det A = -60 \text{ and } \det B = -27,$$

so the second option is definitely easier here. By the theorem, we know that

$$\det(AB) = (\det A)(\det B) = (-60)(-27) = 1620.$$

You should verify that this is the same answer that you would get if you were to first calculate the product AB, then find its determinant.

Determinants and Invertibility

Several sections ago, we introduced the concept of *invertibility*. Recall that a matrix A is *invertible* if there is another matrix, which we denote by A^{-1}, so that

$$AA^{-1} = I.$$

For example, it is easy to see that the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

has inverse

$$A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

In a sense, matrix inverses are the matrix analogue of *real number* multiplicative inverses. Of course, it is quite easy to determine whether or not a real number has an inverse:

$$a \text{ has inverse } \frac{1}{a} \text{ if and only if } a \neq 0.$$

In other words, *every* real number other than 0 has an inverse.

Unfortunately, the question of whether or not a given *square matrix* has an inverse is not quite so simple. Certainly the 0 matrix

$$0_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

has no inverse; however, *many* nonzero matrices also fail to have inverses, such as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So how can we determine whether or not a given square matrix does actually have an inverse? The following theorem tells us that we need merely look at the determinant of the matrix in question:

Theorem. A square matrix A is invertible if and only if $\det A \neq 0$.

In a sense, the theorem says that matrices with determinant 0 act like the *number* 0–they don't have inverses. On the other hand, matrices with nonzero determinants act like all of the other real numbers–they *do* have inverses.

Example

Determine if the following matrices are invertible:

1. $$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2. $$C = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$$

1. As indicated by the theorem, we simply need to look at the determinant:

   $$\det A = \det \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2.$$

   By the theorem, we know that A is invertible.

2. The determinant of C is

   $$\det \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} = 0.$$

   Again using the theorem, we know that C is *not* invertible.

Key Point. Note that, even though the theorem can tell us that the matrix A above has an inverse, we have *no* information as to what matrix A^{-1} actually is. The theorem only tells us that A^{-1} exists.

Determinants of Inverses

Now that we have an easy way to determine whether or not A^{-1} exists by using determinants, we should demand an easy way to calculate $\det(A^{-1})$, when A^{-1} exists. Fortunately, there is an easy way to make the calculation:

Theorem. If A^{-1} exists, then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Proof. Since

$$(\det A)(\det A^{-1}) = \det(AA^{-1}),$$

we have

$$\begin{aligned}(\det A)(\det A^{-1}) &= \det(AA^{-1}) \\ &= \det I \\ &= 1.\end{aligned}$$

Since $(\det A)(\det A^{-1}) = 1$ and $\det A \neq 0$, we see that

$$\det A^{-1} = \frac{1}{\det A},$$

as desired.