# Test1

Thursday, February 25, 2016 7:22 AM

Test1

# Math 300

## Test 1

February 26, 2016

**Name:** Key

You must show ALL of your work in order to receive credit.

If your scratch paper shows work that leads to your solution, please turn in in inside the test. Otherwise, dispose of it yourself.

1. Modified true/false: If the statement below is *always* true, write "true". Otherwise, correct the statement. (2 points each)

   (a) For any $n \times n$ matrices $A$ and $B$, $\det(AB) = \det(A)\det(B)$.

   True

   (b) If $V$ is a vector space, then every element $u$ of $V$ has a unique additive inverse.

   True

   (c) If $A$ is an $n \times n$ matrix with $\det A = c$, $c \neq 0$, then $\det A^\top = \frac{1}{c}$.

   $\det A^{-1} = \frac{1}{c}$ $\boxed{\text{or}}$ $\det A^{T} = c$

   (d) If $A$ and $B$ are matrices so that both products $AB$ and $BA$ are defined, then $A$ and $B$ must both be $n \times n$ matrices.

   $A - m \times n$, $B - n \times m$

   (e) If $A$, $B$, and $C$ are invertible $n \times n$ matrices so that

   $$BAC = CB,$$

   then

   $$A = I.$$

   invertible commuting matrices.

   or, $A = B^{-1}CBC^{-1}$

2. The matrix

$$A = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \\ -3 & 0 & 0 \end{pmatrix}$$

is invertible. Find $A^{-1}$. (15 points)

$$\left(\begin{array}{ccc|ccc} 0 & 1 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} -3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{3} \\ 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3. Given the system

$$\begin{aligned} w + x + z &= 4 \\ x - 2y - z &= 3 \\ w - y - z &= 7, \end{aligned}$$

(a) Write the augmented matrix for the system. (5 points)

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & -1 & 3 \\ 1 & 0 & -1 & -1 & 7 \end{array}\right)$$

(b) Use elementary row operations to solve the system. Parameterize any free variables. (10 points)

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & -1 & 3 \\ 1 & 0 & -1 & -1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & -1 & 3 \\ 0 & -1 & -1 & -2 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -2 & -1 & 3 \\ 0 & 0 & -3 & -3 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}$$

$z$ is free,

$z = t$

$w = 5$

$x = -1 - t$

$y = -2 - t$

4. The augmented matrix for a system is given by

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 1 & -3 & 1 \\ 0 & 0 & a & 1 & 1 \\ 0 & 0 & 0 & b & 2 \end{array}\right).$$

Find values for $a$ and $b$ so that the system satisfies the conditions below, and briefly justify your choices. (5 points each)

(a) The system has no solution.

$b = 0$ → get equation No solution!

$a$ = any nonzero $0x_1 + 0x_2 + 0x_3 + 0x_4 = 2$

(b) The system has exactly one solution.

$a$ = any nonzero
$b$ = any nonzero

Then determinant of coeff. matrix is nonzero, so unique solution.

(c) The system has infinitely many solutions.

$a = 0$, $b = 2$

→ system is consistent and $x_3$ is free.

5. Let $\mathrm{SL}(2,\mathbb{R})$ be the set of all $2\times 2$ matrices $X$ with $\det X = 1$. Clearly SL is a subset of the vectors from the vector space $\mathcal{M}_2(\mathbb{R})$. Is $\mathrm{SL}(2,\mathbb{R})$ a *subspace* of $\mathcal{M}_2(\mathbb{R})$ under the normal operations of matrix addition and scalar multiplication? If so, prove it. If not, show via a specific counterexample. (15 points)

No: $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin SL(2,\mathbb{R})$

since $\det \vec{0} \neq 1$

In addition -

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in SL(2,\mathbb{R})$

But $\begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \notin SL$ since

$\det = \frac{9}{2} \neq 1$

6. Recall that $\mathcal{P}_4(\mathbb{R})$ is the vector space of all polynomials of degree at most 4 with real coefficients; the operations are the usual polynomial addition and scalar multiplication, and the field is $\mathbb{R}$.

(a) Let $U_1$ be the subspace of $\mathcal{P}_4$ of polynomials of the form $\alpha x^2 + \beta$, $\alpha$, $\beta \in \mathbb{R}$, and let $U_2$ be the subspace of all polynomials $f$ of degree less than or equal to 1 so that $f(0) = 0$. Find a vector that is *not* in either $U_1$ or $U_2$ but that *is* in $U_1 + U_2$. (5 points)

Any vector of form $\alpha x^2 + \beta x + \gamma$ is in $U_1 + U_2$, but $3x+5 \notin U_1$ or $U_2$.

(b) Find a vector in $\mathcal{P}_4$ that is *not* in $U_1 + U_2$. (5 points)

any vector of form

$\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \lambda$

with $\alpha \neq 0$ or $\beta \neq 0$

is in $P_4 - U_1 + U_2$.

For example $3x^4 - x^3$.

Form of a vector in $U_1 + U_2$:

$\alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$

7. Let $V$ be a vector space, and let $U_1$ and $U_2$ be subspaces of $V$. The notation $U_1 \cap U_2$ indicates the set of vectors from $V$ that are in both $U_1$ and in $U_2$, that is

$$U_1 \cap U_2 = \{u | u \in U_1 \text{ and } u \in U_2\}.$$

Prove that $U_1 \cap U_2$ is a subspace of $V$. (15 points)

To prove that $U_1 \cap U_2$ is a subspace, we need to show:

(1) $\vec{0} \in U_1 \cap U_2$: $\vec{0} \in U_1$ and $\vec{0} \in U_2$ since both are subspaces, so $\vec{0} \in U_1 \cap U_2$.

(2) $u, v \in U_1 \cap U_2 \Rightarrow u+v \in U_1 \cap U_2$:
If $u, v \in U_1 \cap U_2$, then $u+v \in U_1$ since $u, v$ are and $U_1$ a subspace; similarly, $u+v \in U_2$. So $u+v \in U_1 \cap U_2$.

(3) $\lambda \in \mathbb{F}$, $u \in U_1 \cap U_2 \Rightarrow \lambda u \in U_1 \cap U_2$:
$u \in U_1$, $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U_1$ since $U_1$ a subspace; similarly, $\lambda u \in U_2$. So $\lambda u \in U_1 \cap U_2$.