1. In the first part of this homework, we considered the matrix $A - \lambda I$, where $A$ and $I$ are both $n \times n$, and $\lambda$ is a real or complex variable. We also considered the determinant of this matrix, $\det(A - \lambda I)$; since $\lambda$ is a variable, $\det(A - \lambda I)$ is a polynomial.

In this homework, we will consider the geometric properties of some closely related quantities. Let $A$ be a (fixed) $n \times n$ matrix with entries in $\mathbb{F}$, and let $\lambda$ be any (fixed) number in $\mathbb{F}$. Consider the set of all vectors $x \in \mathbb{F}^n$ so that

$$Ax = \lambda x.$$ 

(a) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\lambda = 3$. Find a vector $x$ in $\mathbb{R}^2$ so that

$$Ax = \lambda x.$$

**Example:** We want to guarantee that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$ 

To understand this in more detail, let us consider the equivalent linear system:

$$2x_1 + x_2 = 3x_1$$

$$x_1 + 2x_2 = 3x_2,$$

or

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0.$$ 

Thus we write the augmented matrix for this equation and solve using row operations:

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

We see that $x_2$ is free, so we choose, say $x_2 = 1$; then $x_1 = 1$ as well, and

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a vector so that

$$Ax = 3x.$$
(b) With $A$ from part (a) and $\lambda = 4$, find the set of all vectors $x$ in $\mathbb{R}^2$ so that $Ax = \lambda x$.

Solution: Using similar reasoning as in the above, we see that we can solve by reducing the augmented matrix

$$
\begin{pmatrix}
-2 & 1 & | & 0 \\
1 & -2 & | & 0
\end{pmatrix}.
$$

Let’s apply row operations:

$$
\begin{pmatrix}
-2 & 1 & | & 0 \\
1 & -2 & | & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -1/2 & | & 0 \\
0 & -3/2 & | & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -1/2 & | & 0 \\
0 & 1 & | & 0
\end{pmatrix}.
$$

In this case, we see that we must choose $x_2 = 0$, so that $x_1 = 0$ as well. Thus the set of all vectors $x \in \mathbb{R}^2$ so that $Ax = 4x$ is simply $\{0\}$.

(c) Let

$$
A = \begin{pmatrix}
0 & -1 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
$$

and $\lambda = i$. Find a vector $x$ in $\mathbb{C}^3$ so that $Ax = \lambda x$.

Example: We want to solve the system whose augmented matrix is given by

$$
\begin{pmatrix}
-i & -1 & 1 & | & 0 \\
1 & 1-i & 1 & | & 0 \\
0 & 1 & 1-i & | & 0
\end{pmatrix}.
$$

Let’s apply row operations:
Thus $x_3$ is free; choosing $x_3 = 1$, we see that

$$x_1 = -1 - 2i \text{ and } x_2 = i - 1,$$

so that

$$x = \begin{pmatrix} -1 - 2i \\ i - 1 \\ 1 \end{pmatrix}$$

is a vector so that $Ax = ix$.

(d) With $A$ and $\lambda$ from part (c), find the set of all vectors $x \in \mathbb{C}^3$ so that

$$Ax = \lambda x.$$

_Solution:_ Working from the example above, we parameterize $x_3$ as $x_3 = t$ so that

$$x = \begin{pmatrix} (-1 - 2i)t \\ (i - 1)t \\ t \end{pmatrix},$$

then $Ax = ix$ if and only if $x$ is of the form above.

2. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x$$

is a subspace of $\mathbb{F}^n$.

_Solution:_ Let $U$ be the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x.$$
Homework 4, Part 2 Key

(a) \( \mathbf{0} \in U \): Clearly \( A\mathbf{0} = \mathbf{0} = \lambda \mathbf{0} \).

(b) If \( x, y \in U \), then \( x + y \in U \): Using properties of matrix arithmetic, we have

\[
A(x + y) = Ax + Ay = \lambda x + \lambda y = \lambda(x + y),
\]

so that \( x + y \in U \).

(c) If \( x \in U \), \( \alpha \in \mathbb{F} \), then \( \alpha x \in U \): Similar to the previous part, we see that

\[
A(\alpha x) = \alpha(Ax) = \alpha(\lambda x) = \lambda(\alpha x),
\]

again using properties of matrix arithmetic, and the fact that multiplication in \( \mathbb{F} \) commutes. Thus \( \alpha x \in U \), and \( U \) is a subspace of \( \mathbb{F}^n \).

3. Given fixed \( A \in \mathcal{M}_n(\mathbb{F}) \) and fixed \( \lambda \in \mathbb{F} \), show that the set of all vectors \( x \in \mathbb{F}^n \) satisfying \( Ax = \lambda x \)

is precisely the set of all solutions to the matrix equation

\[
(A - \lambda I)x = \mathbf{0}.
\]

Solution: Using properties of matrix arithmetic, we see that

\[
Ax = \lambda x \iff Ax - \lambda x = \mathbf{0} \\
\iff Ax - \lambda Ix = \mathbf{0} \\
\iff (A - \lambda I)x = \mathbf{0},
\]

which is true if and only if \( x \) is a solution to the matrix equation.

4. Let \( L \) be a line in \( \mathbb{R}^2 \) that does not pass through the origin, and let \( U \) be the set of all points on \( L \). Does \( U \) form a subspace of \( \mathbb{R}^2 \) under the usual definition of vector addition and scalar multiplication? Prove your claim.

Solution: The set \( U \) never forms a subspace, because it can never contain the \( \mathbf{0} \) vector. For example, if

\[
U = \left\{ \begin{pmatrix} x \\ 3x + 5 \end{pmatrix} \mid x \in \mathbb{R} \right\},
\]

then \( \mathbf{0} \not\in U \); of course, \( U \) is not closed under vector addition or scalar multiplication either. For example,

\[
\begin{pmatrix} 2 \\ 11 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} \not\in U.
\]
5. Let $V$ be a vector space, and let $S$, $T$, and $U$ be subspaces of $V$ so that

$$V = S \oplus U \text{ and } V = T \oplus U;$$

that is, every vector in $V$ may be decomposed uniquely as a sum of a vector from $S$ and a vector from $U$, and every vector in $V$ may be decomposed uniquely as a sum of a vector from $T$ and a vector from $U$. Does it follow that $S = T$? If so, prove it. If not, find a counterexample.

Counterexample: Consider $\mathbb{R}^3$ with its usual operations, and subspaces

$$U = \left\{ \begin{pmatrix} u \\ w \\ 0 \end{pmatrix} \mid u, w \in \mathbb{R} \right\},$$

$$S = \left\{ \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} \mid s \in \mathbb{R} \right\},$$

and

$$T = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Now it is clear that:

(a) $S \neq T$, and

(b) $S \cap U = \{0\} = T \cap U$, so that the sums are direct, and

(c) $\mathbb{R}^3 = S \oplus U$, $\mathbb{R}^3 = T \oplus U$.

6. Let $U$ be the subspace of $\mathbb{R}^4$ defined by

$$U = \left\{ \begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}.$$

Find a subspace $W$ of $\mathbb{R}^4$ so that

$$\mathbb{R}^4 = U \oplus W.$$

Be sure to show that every vector in $\mathbb{R}^4$ can be written as a sum of vectors in $U$ and $W$.

Example: Set

$$W = \left\{ \begin{pmatrix} t \\ 0 \\ s \\ 0 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.$$
Now it is clear that $U \cap W = \{0\}$, since if
\[
\begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ s \\ 0 \end{pmatrix},
\]
then we must have $x = 0$ and $y = 0$, so that $t = s = 0$ as well.
Thus
\[
U + W = U \oplus W
\]
is a direct sum; it remains to show that the sum is all of $\mathbb{R}^4$.
Let
\[
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]
be any vector in $\mathbb{R}^4$. Choose $x = b$, $y = d$, $t = -b + a$, and $s = -d + c$. Then
\[
\begin{pmatrix} b \\ b \\ d \\ d \end{pmatrix} \in U \text{ and } \begin{pmatrix} -b + a \\ 0 \\ -d + c \\ 0 \end{pmatrix} \in W,
\]
and
\[
\begin{pmatrix} b \\ b \\ d \\ d \end{pmatrix} + \begin{pmatrix} -b + a \\ 0 \\ -d + c \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},
\]
as desired. Thus
\[
\mathbb{R}^4 = U \oplus W.
\]