1. In the first part of this homework, we considered the matrix $A - \lambda I$, where A and I are both $n \times n$, and λ is a real or complex variable. We also considered the determinant of this matrix, $\det(A - \lambda I)$; since λ is a variable, $\det(A - \lambda I)$ is a polynomial.

   In this homework, we will consider the geometric properties of some closely related quantites. Let A be a (fixed) $n \times n$ matrix with entries in $\mathbb{F}$, and let λ be any (fixed) number in $\mathbb{F}$. Consider the set of all vectors $x \in \mathbb{F}^n$ so that

   $$Ax = \lambda x.$$

   (a) Let

   $$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   and $\lambda = 3$. Find a vector x in $\mathbb{R}^2$ so that

   $$Ax = \lambda x.$$

   Example: We want to guarantee that

   $$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

   To understand this in more detail, let us consider the equivalent linear system:

   $$\begin{aligned} 2x_1 + x_2 &= 3x_1 \\ x_1 + 2x_2 &= 3x_2, \end{aligned}$$

   or

   $$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0. \end{aligned}$$

   Thus we write the augmented matrix for this equation and solve using row operations:

   $$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

   We see that x_2 is free, so we choose, say $x_2 = 1$; then $x_1 = 1$ as well, and

   $$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   is a vector so that

   $$Ax = 3x.$$

(b) With A from part (a) and $\lambda = 4$, find the set of *all* vectors x in $\mathbb{R}^2$ so that $Ax = \lambda x$.

Solution: Using similar reasoning as in the above, we see that we can solve by reducing the augmented matrix

$$\begin{pmatrix} -2 & 1 & \mid & 0 \\ 1 & -2 & \mid & 0 \end{pmatrix}.$$

Let's apply row operations:

$$\begin{aligned} \begin{pmatrix} -2 & 1 & \mid & 0 \\ 1 & -2 & \mid & 0 \end{pmatrix} &\to \begin{pmatrix} 1 & -1/2 & \mid & 0 \\ 1 & -2 & \mid & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -1/2 & \mid & 0 \\ 0 & -3/2 & \mid & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & -1/2 & \mid & 0 \\ 0 & 1 & \mid & 0 \end{pmatrix}. \end{aligned}$$

In this case, we see that we *must* choose $x_2 = 0$, so that $x_1 = 0$ as well. Thus the set of all vectors $x \in \mathbb{R}^2$ so that $Ax = 4x$ is simply $\{\mathbf{0}\}$.

(c) Let

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and $\lambda = i$. Find a vector x in $\mathbb{C}^3$ so that

$$Ax = \lambda x.$$

Example: We want to solve the system whose augmented matrix is given by

$$\begin{pmatrix} -i & -1 & 1 & \mid & 0 \\ 1 & 1-i & 1 & \mid & 0 \\ 0 & 1 & 1-i & \mid & 0 \end{pmatrix}.$$

Let's apply row operations:

$$\begin{pmatrix} -i & -1 & 1 & | & 0 \\ 1 & 1-i & 1 & | & 0 \\ 0 & 1 & 1-i & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -i & i & | & 0 \\ 1 & 1-i & 1 & | & 0 \\ 0 & 1 & 1-i & | & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -i & i & | & 0 \\ 0 & 1 & 1-i & | & 0 \\ 0 & 1 & 1-i & | & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -i & i & | & 0 \\ 0 & 1 & 1-i & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 2i+1 & | & 0 \\ 0 & 1 & 1-i & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Thus x_3 is free; choosing $x_3 = 1$, we see that

$$x_1 = -1 - 2i \text{ and } x_2 = i - 1,$$

so that

$$x = \begin{pmatrix} -1-2i \\ i-1 \\ 1 \end{pmatrix}$$

is a vector so that $Ax = ix$.

(d) With A and λ from part (c), find the set of *all* vectors $x \in \mathbb{C}^3$ so that

$$Ax = \lambda x.$$

Solution: Working from the example above, we parameterize x_3 as $x_3 = t$ so that

$$x = \begin{pmatrix} (-1-2i)t \\ (i-1)t \\ t \end{pmatrix};$$

then $Ax = ix$ if and only if x is of the form above.

2. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x$$

is a subspace of $\mathbb{F}^n$.

Solution: Let U be the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x.$$

We must show that:

(a) $\mathbf{0} \in U$: Clearly $A\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$.

(b) If $x, y \in U$, then $x + y \in U$: Using properties of matrix arithmetic, we have

$$\begin{aligned} A(x+y) &= Ax + Ay \\ &= \lambda x + \lambda y \\ &= \lambda(x+y), \end{aligned}$$

so that $x + y \in U$.

(c) If $x \in U$, $\alpha \in \mathbb{F}$, then $\alpha x \in U$: Similar to the previous part, we see that

$$\begin{aligned} A(\alpha x) &= \alpha(Ax) \\ &= \alpha(\lambda x) \\ &= \lambda(\alpha x), \end{aligned}$$

again using properties of matrix arithmetic, and the fact that multiplication in $\mathbb{F}$ commutes. Thus $\alpha x \in U$, and U is a subspace of $\mathbb{F}^n$.

3. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x$$

is precisely the set of all solutions to the matrix equation

$$(A - \lambda I)x = \mathbf{0}.$$

Solution: Using properties of matrix arithmetic, we see that

$$\begin{aligned} Ax = \lambda x &\iff Ax - \lambda x = \mathbf{0} \\ &\iff Ax - \lambda I x = \mathbf{0} \\ &\iff (A - \lambda I)x = \mathbf{0}, \end{aligned}$$

which is true if and only if x is a solution to the matrix equation.

4. Let L be a line in $\mathbb{R}^2$ that *does not* pass through the origin, and let U be the set of all points on L. Does U form a subspace of $\mathbb{R}^2$ under the usual definition of vector addition and scalar multplication? Prove your claim.

Solution: The set U *never* forms a subspace, because it can never contain the $\mathbf{0}$ vector. For example, if

$$U = \left\{ \begin{pmatrix} x \\ 3x+5 \end{pmatrix} | x \in \mathbb{R} \right\},$$

then $\mathbf{0} \notin U$; of course, U is not closed under vector addition or scalar multiplication either. For example,

$$\begin{pmatrix} 2 \\ 11 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} \notin U.$$

5. Let V be a vector space, and let S, T, and U be subspaces of V so that

$$V = S \oplus U \text{ and } V = T \oplus U;$$

that is, every vector in V may be decomposed uniquely as a sum of a vector from S and a vector from U, and every vector in V may be decomposed uniquely as a sum of a vector from T and a vector from U. Does it follow that $S = T$? If so, prove it. If not, find a counterexample.

Counterexample: Consider $\mathbb{R}^3$ with its usual operations, and subspaces

$$U = \left\{ \begin{pmatrix} u \\ w \\ 0 \end{pmatrix} | u, w \in \mathbb{R} \right\},$$

$$S = \left\{ \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} | s \in \mathbb{R} \right\}, \text{ and}$$

$$T = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} | t \in \mathbb{R} \right\}.$$

Now it is clear that:

(a) $S \neq T$, and

(b) $S \cap U = \{\mathbf{0}\} = T \cap U$, so that the sums are direct, and

(c) $\mathbb{R}^3 = S \oplus U$, $\mathbb{R}^3 = T \oplus U$.

6. Let U be the subspace of $\mathbb{R}^4$ defined by

$$U = \left\{ \begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} | x, y \in \mathbb{R} \right\}.$$

Find a subspace W of $\mathbb{R}^4$ so that

$$\mathbb{R}^4 = U \oplus W.$$

Be sure to show that every vector in $\mathbb{R}^4$ can be written as a sum of vectors in U and W.

Example: Set

$$W = \left\{ \begin{pmatrix} t \\ 0 \\ s \\ 0 \end{pmatrix} | t, s \in \mathbb{R} \right\}.$$

Now it is clear that $U \cap W = \{\mathbf{0}\}$, since if

$$\begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ s \\ 0 \end{pmatrix},$$

then we must have $x = 0$ and $y = 0$, so that $t = s = 0$ as well.

Thus

$$U + W = U \oplus W$$

is a direct sum; it remains to show that the sum is all of $\mathbb{R}^4$.

Let

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

be any vector in $\mathbb{R}^4$. Choose $x = b$, $y = d$, $t = -b + a$, and $s = -d + c$. Then

$$\begin{pmatrix} b \\ b \\ d \\ d \end{pmatrix} \in U \text{ and } \begin{pmatrix} -b + a \\ 0 \\ -d + c \\ 0 \end{pmatrix} \in W,$$

and

$$\begin{pmatrix} b \\ b \\ d \\ d \end{pmatrix} + \begin{pmatrix} -b + a \\ 0 \\ -d + c \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

as desired. Thus

$$\mathbb{R}^4 = U \oplus W.$$