1. In the first part of this homework, we considered the matrix $A - \lambda I$, where A and I are both $n \times n$, and λ is a real or complex variable. We also considered the determinant of this matrix, $\det(A - \lambda I)$; since λ is a variable, $\det(A - \lambda I)$ is a polynomial.

   In this homework, we will consider the geometric properties of some closely related quantites. Let A be a (fixed) $n \times n$ matrix with entries in $\mathbb{F}$, and let λ be any (fixed) number in $\mathbb{F}$. Consider the set of all vectors $x \in \mathbb{F}^n$ so that

   $$Ax = \lambda x.$$

   (a) Let

   $$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   and $\lambda = 3$. Find a vector x in $\mathbb{R}^2$ so that

   $$Ax = \lambda x.$$

   (b) With A from part (a) and $\lambda = 4$, find the set of *all* vectors x in $\mathbb{R}^2$ so that $Ax = \lambda x$.

   (c) Let

   $$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   and $\lambda = i$. Find a vector x in $\mathbb{C}^3$ so that

   $$Ax = \lambda x.$$

   (d) With A and λ from part (c), find the set of *all* vectors $x \in \mathbb{C}^3$ so that

   $$Ax = \lambda x.$$

2. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

   $$Ax = \lambda x$$

   is a subspace of $\mathbb{F}^n$.

3. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

   $$Ax = \lambda x$$

   is precisely the set of all solutions to the matrix equation

   $$(A - \lambda I)x = \mathbf{0}.$$

4. Let L be a line in $\mathbb{R}^2$ that *does not* pass through the origin, and let U be the set of all points on L. Does U form a subspace of $\mathbb{R}^2$ under the usual definition of vector addition and scalar multplication? Prove your claim.

5. Let V be a vector space, and let S, T, and U be subspaces of V so that

$$V = S \oplus U \text{ and } V = T \oplus U;$$

that is, every vector in V may be decomposed uniquely as a sum of a vector from S and a vector from U, and every vector in V may be decomposed uniquely as a sum of a vector from T and a vector from U. Does it follow that $S = T$? If so, prove it. If not, find a counterexample.

6. Let U be the subspace of $\mathbb{R}^4$ defined by

$$U = \left\{ \begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} | x, y \in \mathbb{R} \right\}.$$

Find a subspace W of $\mathbb{R}^4$ so that

$$\mathbb{R}^4 = U \oplus W.$$

Be sure to show that every vector in $\mathbb{R}^4$ can be written as a sum of vectors in U and W.