1. In the first part of this homework, we considered the matrix $A - \lambda I$, where $A$ and $I$ are both $n \times n$, and $\lambda$ is a real or complex variable. We also considered the determinant of this matrix, $\det(A - \lambda I)$; since $\lambda$ is a variable, $\det(A - \lambda I)$ is a polynomial.

In this homework, we will consider the geometric properties of some closely related quantities. Let $A$ be a (fixed) $n \times n$ matrix with entries in $\mathbb{F}$, and let $\lambda$ be any (fixed) number in $\mathbb{F}$. Consider the set of all vectors $x \in \mathbb{F}^n$ so that

$$Ax = \lambda x.$$ 

(a) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\lambda = 3$. Find a vector $x$ in $\mathbb{R}^2$ so that

$$Ax = \lambda x.$$ 

(b) With $A$ from part (a) and $\lambda = 4$, find the set of all vectors $x$ in $\mathbb{R}^2$ so that $Ax = \lambda x$.

(c) Let

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and $\lambda = i$. Find a vector $x$ in $\mathbb{C}^3$ so that

$$Ax = \lambda x.$$ 

(d) With $A$ and $\lambda$ from part (c), find the set of all vectors $x \in \mathbb{C}^3$ so that

$$Ax = \lambda x.$$ 

2. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x$$

is a subspace of $\mathbb{F}^n$.

3. Given fixed $A \in \mathcal{M}_n(\mathbb{F})$ and fixed $\lambda \in \mathbb{F}$, show that the set of all vectors $x \in \mathbb{F}^n$ satisfying

$$Ax = \lambda x$$

is precisely the set of all solutions to the matrix equation

$$(A - \lambda I)x = 0.$$ 

4. Let $L$ be a line in $\mathbb{R}^2$ that does not pass through the origin, and let $U$ be the set of all points on $L$. Does $U$ form a subspace of $\mathbb{R}^2$ under the usual definition of vector addition and scalar multiplication? Prove your claim.
5. Let $V$ be a vector space, and let $S$, $T$, and $U$ be subspaces of $V$ so that

$$V = S \oplus U \text{ and } V = T \oplus U;$$

that is, every vector in $V$ may be decomposed uniquely as a sum of a vector from $S$ and a vector from $U$, and every vector in $V$ may be decomposed uniquely as a sum of a vector from $T$ and a vector from $U$. Does it follow that $S = T$? If so, prove it. If not, find a counterexample.

6. Let $U$ be the subspace of $\mathbb{R}^4$ defined by

$$U = \left\{ \begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}. $$

Find a subspace $W$ of $\mathbb{R}^4$ so that

$$\mathbb{R}^4 = U \oplus W.$$

Be sure to show that every vector in $\mathbb{R}^4$ can be written as a sum of vectors in $U$ and $W$. 