1. Let $\lambda$ be a variable which takes on real or complex values, and consider the matrix $A - \lambda I$, where $A$ and $I$ are both $n \times n$. Calculate the matrix $A - \lambda I$ and its determinant $\det(A - \lambda I)$ for each of the following choices of $A$:

   (a) $A = (4)$ *Solution:* $A - \lambda I$ is the matrix $\begin{pmatrix} 4 - \lambda \end{pmatrix}$, and its determinant is

   $$\det(A - \lambda I) = 4 - \lambda.$$

   (b) $A = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}$
   *Solution:* The matrix $A - \lambda I$ is given by

   $$A - \lambda I = \begin{pmatrix} 2 - \lambda & 1 \\ 4 & 1 - \lambda \end{pmatrix}$$

   and its determinant is

   $$\begin{aligned} \det(A - \lambda I) &= (2 - \lambda)(1 - \lambda) - 4 \\ &= 2 - 3\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 3\lambda - 2. \end{aligned}$$

   (c) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
   *Solution:* The matrix $A - \lambda I$ is given by

   $$A - \lambda I = \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1 - \lambda & 0 \\ -1 & 0 & -\lambda \end{pmatrix}$$

   and its determinant is

   $$\begin{aligned} \det(A - \lambda I) &= -\lambda(-\lambda(1 - \lambda)) + (-(-1(1 - \lambda))) \\ &= \lambda^2 - \lambda^3 + 1 - \lambda \\ &= -\lambda^3 + \lambda^2 - \lambda + 1. \end{aligned}$$

2. Given an $n \times n$ matrix $A$, set $p(\lambda) = \det(A - \lambda I)$. Calculate $p(0)$ for each of the matrices $A$ from part (1), and determine what information $p(0)$ provides about $A$.

   (a) *Solution:* $p(\lambda) = 4 - \lambda$, so $p(0) = 4$.

   (b) *Solution:* $p(\lambda) = \lambda^2 - 3\lambda - 2$, so $p(0) = -2$.

(c) *Solution:* $p(\lambda) = -\lambda^3 + \lambda^2 - \lambda + 1$, so $p(0) = 1$. In each example, $p(0)$ is the determinant of the original matrix $A$.

3. Consider the set $\mathbb{R}_+$ of all positive real numbers, with the following definitions for addition and scalar multiplication (to avoid confusion, we use the symbol $\square$ to refer to the addition operation, and $\wedge$ to refer to the operation of scalar multiplication):

$$a \square b = ab \text{ and } \lambda \wedge a = a^\lambda.$$

In other words, our "addition" in $\mathbb{R}_+$ is just real number multiplication, and our "scalar multiplication" raises a given vector in $\mathbb{R}_+$ to the designated scalar power.

$\mathbb{R}_+$ together with these operations is a vector space over $\mathbb{R}$.

(a) Calculate $3 \square 5$.
*Solution:* $3 \square 5 = 15$.

(b) Calculate $-2 \wedge 3$.
*Solution:* $-2 \wedge 3 = 3^{-2} = 1/9$.

(c) Show that $\mathbb{R}_+$ is closed under scalar multiplication by elements of $\mathbb{R}$.
*Solution:* Given any positive number $a$ and any $\lambda \in \mathbb{R}$, $a^\lambda > 0$, so that $\lambda \wedge a \in \mathbb{R}_+$.

(d) What number serves as the additive identity in $\mathbb{R}_+$?
*Solution:* 1 is the additive identity, since

$$1 \square a = 1 \cdot a = a$$

for all $a \in \mathbb{R}_+$.

(e) Given an element $a$ of $\mathbb{R}_+$, determine the form of the additive inverse of $a$.
*Solution:* The additive inverse of $a \in \mathbb{R}_+$ is $1/a$, since

$$a \square \frac{1}{a} = a \frac{1}{a} = 1.$$

(f) $\mathbb{R}_+$ is a subset of elements of $\mathbb{R}$. Is $\mathbb{R}_+$ with the operations defined above a subspace of $\mathbb{R}$ (under the normal operations on $\mathbb{R}$)? Explain why/why not.
*Solution:* $\mathbb{R}_+$ is not a subspace of $\mathbb{R}$, as the operations of addition and scalar multiplication are different.