1. Use cofactor expansion to calculate the determinant of the matrix

$$A = \begin{pmatrix} 0 & 4 & 3 & 1 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix}.$$

Include *all* of your work.

*Solution:*

$$\begin{aligned}
\det(A) &= \det \begin{pmatrix} 0 & 4 & 3 & 1 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix} \\
&= -3 \det \begin{pmatrix} 0 & 3 & 1 & -2 \\ 2 & 3 & -1 & 0 \\ 3 & 2 & -5 & 1 \\ 1 & -4 & 0 & 0 \end{pmatrix} + 2 \det \begin{pmatrix} 0 & 4 & 3 & 1 \\ 2 & 2 & 3 & -1 \\ 3 & 1 & 2 & -5 \\ 1 & 0 & -4 & 0 \end{pmatrix} \\
&= -3\Big( - \det \begin{pmatrix} 3 & 1 & -2 \\ 3 & -1 & 0 \\ 2 & -5 & 1 \end{pmatrix} - 4 \det \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 0 \\ 3 & -5 & 1 \end{pmatrix}\Big) \\
&\quad +2\Big( - \det \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & -5 \end{pmatrix} + 4 \det \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & -5 \end{pmatrix}\Big) \\
&= -3\Big(-\big( - 2(-15+2) + (-3-3)\big) - 4\big( - (2+0) - 2(-10+3)\big)\Big) \\
&\quad +2\Big(-\big((4-3) + (8-3) - 5(12-6)\big) + 4\big( - 2(-20-1) + 3(-4-2)\big)\Big) \\
&= -3\Big(-\big(26-6\big) - 4\big( - 2 + 14\big)\Big) + 2\Big(-\big(1+5-30\big) + 4\big(42-18\big)\Big) \\
&= -3\Big( - 20 - 48\Big) + 2\Big(24 + 96\Big) \\
&= 204 + 240 \\
&= 444.
\end{aligned}$$

2. Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let $B$ be the matrix obtained from $A$ by multiplying row $k$ of $A$ by the constant $c$. Prove that $\det(B) = c\det(A)$.

*Solution:*

The determinant of $A$ is precisely its cofactor expansion along row $k$, given by

$$\det(A) = \sum_{i=1}^{n} a_{ki} C_{ki}.$$

Recall that $C_{ki}$ is (up to a possible sign change) the determinant of the submatrix of $A$ obtained by deleting row $k$ and column $i$ of $A$.

Now the only row of $B$ which differs from a row of $A$ is the $k$th, so that the cofactors $C'_{ki}$ of $B$ match up with those of $A$ along row $k$, that is

$$C'_{ki} = C_{ki} \ \forall \ i.$$

Thus by expanding along row $k$ of $B$, we see that

$$\begin{aligned} \det(B) &= \sum_{i=1}^{n} c a_{ki} C_{ki} \\ &= c \sum_{i=1}^{n} a_{ki} C_{ki} \\ &= c \det(A). \end{aligned}$$

3. Let $A$ be an $n \times n$ matrix. Use induction to prove that for any constant $c$, $\det(cA) = c^n \det(A)$. **Note:** The determinant of a $1 \times 1$ matrix $A = (a_{11})$ is just $\det(A) = a_{11}$.

   *Solution:*

   The statement is clearly true for $n = 1$: if $A = (a_{11})$, then $\det(A) = a_{11}$ and $\det(cA) = ca_{11}$.

   Assume that the statement is true for all $n \times n$ matrices. Let $A$ be an $(n+1) \times (n+1)$ matrix, and consider the matrix $cA$. We may calculate the determinant of $cA$ by expanding along, say, row 1, that is

   $$\det(cA) = \sum_{i=1}^{n+1} c a_{1i} C'_{1i}.$$

   Now each $C'_{1i}$ is (up to a sign change) the determinant of the $n \times n$ submatrix $cA_{1i}$ of $cA$. By the inductive hypothesis,

   $$\det(cA_{1i}) = c^n \det(A_{1i}),$$

   so that the cofactors $C_{1i}$ of $A$ differ from the cofactors $C'_{1i}$ of $cA$ by a factor of $c^n$, that is

   $$C'_{1i} = c^n C_{1i}.$$

Thus

$$\begin{aligned}
\det(cA) &= \sum_{i=1}^{n+1} ca_{1i}C'_{1i} \\
&= \sum_{i=1}^{n+1} c^{n+1}a_{1i}C_{1i} \\
&= c^{n+1}\sum_{i=1}^{n+1} a_{1i}C_{1i} \\
&= c^{n+1}\det(A).
\end{aligned}$$

4. Prove that, for any square matrix $A$, $\det(A) = \det(A^\top)$.

*Solution:*

We proceed by induction: the theorem is clearly true for a $1 \times 1$ matrix, since $(a_{11})^\top = (a_{11})$.

Assume the theorem is true for any $n \times n$ matrix, and let $A$ be $(n+1) \times (n+1)$. By expanding along row 1 of $A$, we see that

$$\det(A) = \sum_{i=1}^{n} a_{1i}C_{1i}.$$

Now consider the matrices $A_{ij}$ which are obtained from $A$ by deleting row $i$ and column $j$. Each $A_{ij}$ is an $n \times n$ matrix and thus the induction hypothesis applies: $\det(A_{ij}) = \det(A_{ij}^\top)$.

Now consider the submatrix $A'_{ji}$ of $A^\top$ which again is obtained from $A^\top$ by deleting row $j$ and column $i$ of $A^\top$.

I would like to compare the entries of $A$ which appear in $A_{ij}$ to the entries of $A^\top$ which appear in $A'_{ji}$. There are four cases for an entry $a_{km}$ of $A$:

(a) $k < i$, $m < j$: $a_{km}$ appears as the $k, m$ entry of both $A$ and $A_{ij}$; in addition, $a_{km}$ is the $m, k$ entry of $A^\top$, and since $k < i$, $m < j$, $a_{km}$ is also the $m, k$ entry of $A'_{ji}$.

(b) $k > i$, $m < j$: $a_{km}$ is the $k, m$ entry of $A$ but the $k-1, m$ entry of $A_{ij}$; $a_{km}$ is the $m, k$ entry of $A^\top$ and the $m, k-1$ entry of $A'_{ji}$.

(c) $k < i$, $m > j$: $a_{km}$ is the $k, m$ entry of $A$ but the $k, m-1$ entry of $A_{ij}$; $a_{km}$ is the $m, k$ entry of $A^\top$ and the $m-1, k$ entry of $A'_{ji}$.

(d) $k > i$, $m > j$: $a_{km}$ is the $k, m$ entry of $A$ but the $k-1, m-1$ entry of $A_{ij}$; $a_{km}$ is the $m, k$ entry of $A^\top$ and the $m-1, k-1$ entry of $A'_{ji}$.

Now it is clear that $A_{ij}^\top = A'_{ji}$; thus using the inductive hypothesis, we see that the $i, j$ cofactor $C_{ij}$ of $A$ and the $j, i$ cofactor $C'_{ji}$ of $A^\top$ are the same, that is

$$C_{ij} = C'_{ji}.$$

Finally, let us compute the determinant of $A^\top = (a'_{ij})$ directly using cofactor expansion along the first column of $A^\top$. Recall that $a'_{ij} = a_{ji}$, where $a'_{ij}$ is the $i, j$ entry of $A^\top$ and $a_{ji}$ is the $j, i$ entry of $A$:

$$\begin{aligned}\det(A^\top) &= \sum_{i=1}^{n} a'_{i1} C'_{i1} \\ &= \sum_{i=1}^{n} a_{1i} C_{1i} \\ &= \det(A).\end{aligned}$$

5. Prove that, for any square matrix $A$, $\det(A^*) = \overline{\det(A)}$. You may use standard results on complex numbers without proof, as long as the results are stated clearly.

   *Solution:* Recall that, for any complex numbers $a$ and $b$, $\overline{(a+b)} = \overline{a} + \overline{b}$, and $\overline{ab} = \overline{a} \cdot \overline{b}$.

   Now $\det(\overline{A})^\top = \det(\overline{A})$ by problem 4, so we merely need to show that $\det(\overline{A}) = \overline{\det(A)}$.

   We proceed by induction: in fact the argument is nearly identical to that in problems 3 and 4, so we skip to the last step. We see that

$$\det(\overline{A}) = \sum_{i=1}^{n+1} \overline{a_{1i}} C'_{1i},$$

   where $C'_{1i}$ is the $1, i$ cofactor of $\overline{A}$. By the inductive hypothesis, $C'_{1i}$ is just the complex conjugate of the $1, i$ cofactor $C_{1i}$ of $A$, that is

$$C'_{1i} = \overline{C_{1i}}.$$

   Thus we have

$$\begin{aligned}\det(\overline{A}) &= \sum_{i=1}^{n+1} \overline{a_{1i}} C'_{1i} \\ &= \sum_{i=1}^{n+1} \overline{a_{1i}}\,\overline{C_{1i}} \\ &= \sum_{i=1}^{n+1} \overline{a_{1i} C_{1i}} \\ &= \overline{\sum_{i=1}^{n+1} a_{1i} C_{1i}} \\ &= \overline{\det(A)}.\end{aligned}$$

6. An $n \times n$ matrix $A$ (with real or complex entries) is called *skew-symmetric* if $A^\top = -A$. Find a (nonzero) example of a $3 \times 3$ skew-symmetric matrix.

   *Example:*

   Let
$$A = \begin{pmatrix} 0 & i & 1+i \\ -i & 0 & -2 \\ -1-i & 2 & 0 \end{pmatrix}.$$

   Then
$$\begin{aligned} A^\top &= \begin{pmatrix} 0 & -i & -1-i \\ i & 0 & 2 \\ 1+i & -2 & 0 \end{pmatrix} \\ &= -A. \end{aligned}$$

   Thus $A$ is skew-symmetric.

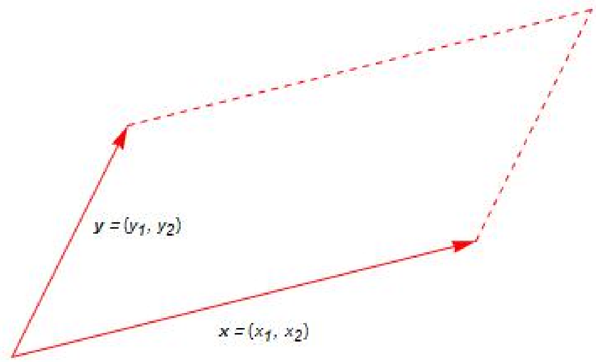
7. Let $A$ be a skew-symmetric matrix.

   (a) Prove that if $n$ is odd, then $\det(A) = 0$.
   *Solution:*
   Clearly $\det(A) = \det(A^\top) = \det(-A)$. Now $n$ is odd, so
$$\begin{aligned} \det(-A) &= (-1)^n \det(A) \\ &= -\det(A). \end{aligned}$$

   Thus we have $\det(A) = -\det(A)$, so that $\det(A) = 0$.

   (b) Find an example of an $n \times n$ skew-symmetric matrix $A$, $n$ even, so that $\det(A) \neq 0$.
   *Example:*
   The matrix
$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$
   is skew-symmetric, and since $i^2 = -1$, its determinant is nonzero.

8. Let $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, where $x_i,\ y_i \in \mathbb{R}$, be nonparallel vectors in the $xy$ plane written in component form (for example, $x$ is the vector starting at the origin and ending at the point $(x_1, x_2)$). Consider the parallelogram below:



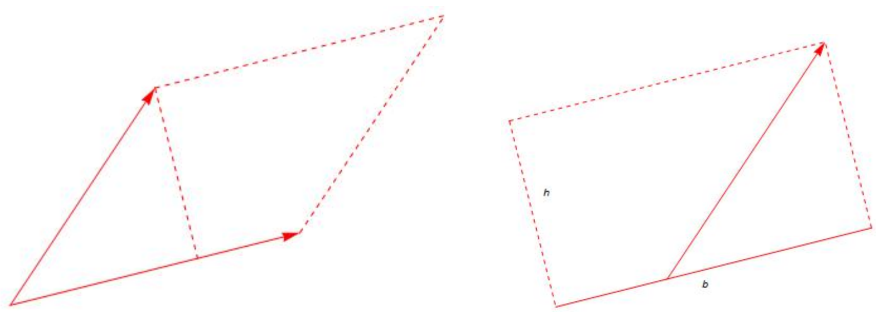

Prove that this parallelogram has area

$$|\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}|.$$

(Hint: use vector projections and other standard calculus results; you may use these without proof, as long as you clearly state the results that you are using).

*Solution:*

We begin by noticing that we can create a rectangle with the same area as the parallelogram by cutting off the right triangle indicated below on the left and moving it to fill in the (equal, due to parallel lines) angle on the right, as indicated below:

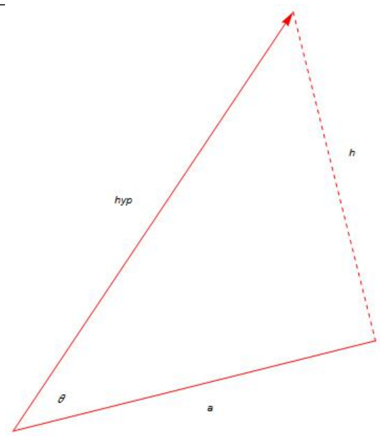



Thus it is clear that the area of the parallelogram is $A = bh$, with $b$ and $h$ indicated in the graphic above.

Now $b$ is just the length of the vector $x = \langle x_1, x_2 \rangle$, that is

$$b = \sqrt{x_1^2 + x_2^2}.$$

To find $h$, we will need to use properties of right triangles; but first, we need to find the length of the adjacent side of the right triangle indicated below:




Now the hypotenuse of the triangle is just the length of the vector $y = \langle y_1, y_2 \rangle$, that is $hyp = \sqrt{y_1^2 + y_2^2}$. We can find the length $a$ of the base of the triangle using projections: the scalar projection of vector $y$ onto vector $x$ is exactly

$$\frac{y \cdot x}{|x|} = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}},$$

where "$\cdot$" is the usual dot product in $\mathbb{R}^2$.

We have a right triangle whose hypotenuse has length

$$\sqrt{y_1^2 + y_2^2}$$

and one of whose legs has length

$$\frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}}.$$

Using the Pythagorean Identity, we see that the length of the remaining leg is

$$\begin{aligned}
\sqrt{\left(\sqrt{y_1^2+y_2^2}\right)^2-\left(\frac{x_1y_1+x_2y_2}{\sqrt{x_1^2+x_2^2}}\right)^2} &= \sqrt{y_1^2+y_2^2-\frac{(x_1y_1+x_2y_2)^2}{x_1^2+x_2^2}} \\
&= \sqrt{\frac{(x_1^2+x_2^2)(y_1^2+y_2^2)-(x_1y_1+x_2y_2)^2}{x_1^2+x_2^2}} \\
&= \sqrt{\frac{x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2-(x_1y_1+x_2y_2)^2}{x_1^2+x_2^2}} \\
&= \sqrt{\frac{x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2-(x_1^2y_1^2+2x_1x_2y_1y_2+x_2^2y_2^2)}{x_1^2+x_2^2}} \\
&= \sqrt{\frac{x_1^2y_2^2-2x_1x_2y_1y_2+x_2^2y_1^2}{x_1^2+x_2^2}} \\
&= \sqrt{\frac{(x_1y_2-x_2y_1)^2}{x_1^2+x_2^2}} \\
&= \frac{|x_1y_2-x_2y_1|}{\sqrt{x_1^2+x_2^2}}
\end{aligned}$$

Thus the area of the original rectangle is

$$A=\frac{|x_1y_2-x_2y_1|}{\sqrt{x_1^2+x_2^2}}\cdot\sqrt{x_1^2+x_2^2}=|x_1y_2-x_2y_1|,$$

which is precisely the absolute value of the determinant of the matrix

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

9. A matrix $A$ is called *orthogonal* if $A^\top = A^{-1}$.

   (a) Find an example of a $2\times 2$ orthogonal matrix (other than $I$) with all real entries.
   *Solution:*
   The matrix

   $$A=\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

   is one example; using the formula for the inverse of an invertible $2\times 2$ matrix,

   $$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we see that

$$\begin{aligned} A^{-1} &= \frac{1}{\frac{3}{4}+\frac{1}{4}} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= A^\top, \end{aligned}$$

(b) Prove that if $A$ is orthogonal, then $\det(A) = \pm 1$.
*Solution:* If $A$ is orthogonal, then $A^{-1} = A^\top$ implies that

$$\begin{aligned} \det(A) &= \det(A^\top) \\ &= \det(A^{-1}) \\ &= \frac{1}{\det(A)}. \end{aligned}$$

In particular,

$$\det(A) = \frac{1}{\det(A)} \implies (\det(A))^2 = 1,$$

so $\det(A) = \pm 1$.

(c) **Bonus:** Prove that a $2 \times 2$ real orthogonal matrix $A$ with $\det(A) = 1$ has form

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $\theta$ is any real number.
*Solution:* If matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $2 \times 2$ real orthogonal with $\det A = 1$, then we know that

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ and } A^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

are equal. Thus we know that

$$\begin{aligned} a &= d \\ b &= -c. \end{aligned}$$

Combined with the fact that $ad - bc = 1$, we see that $a$ and $b$ are any numbers so that

$$a^2 + b^2 = 1.$$

Of course, this is just a description of all points on the unit circle, so we may parameterize $a$ as $a = \cos\theta$ and $b$ as $b = \sin\theta$. Thus any $2 \times 2$ real orthogonal determinant 1 matrix $A$ has form

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

10. Find an example of a system of 3 equations in 3 unknowns that has no solutions. Use geometric reasoning to explain why the system has no solution.

*Solution:* The system

$$\begin{aligned} 10x + 2y - 2z &= 5 \\ 5x + y - z &= 1 \\ x + y + z &= 1 \end{aligned}$$

has no solution. Each equation describes a plane in $\mathbb{R}^3$. The planes described by the first and second equations have normals

$$\langle 10, 2, -2\rangle \text{ and } \langle 5, 1, -1\rangle,$$

respectively. The normal vectors are parallel, which means that the planes are parallel, and it is easy to see that they are different planes, thus never intersect. So regardless of the last equation, there can be no solution to the system: there is no point that lies on both of the first planes, so no point that lies on all three.

11. Use Gauss-Jordan elimination to solve the system

$$\begin{aligned} 6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 &= -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 &= 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 &= 33. \end{aligned}$$

Include all of your work in your solution. If there is no solution, write "inconsistent." If there are infinitely many solutions, paramaterize all free variables in the solution.

*Solution:*

The system has augmented matrix

$$\begin{pmatrix} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{pmatrix}.$$

Let's reduce the system using Gauss-Jordan elimination:

$$\begin{pmatrix} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & 1/2 & -1 & 0 & 5/2 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & 1/2 & -1 & 0 & 5/2 \\ 0 & 0 & -1/3 & 2/3 & 1 & -7/3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & -1/3 & 2/3 & 1 & -7/3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1/3 & -2/3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -5/6 & 8/3 & 0 & -49/6 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & -2 & 0 & | & 5 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

The system is now in RREF. Note that $x_2$ and $x_4$ are free variables; we parameterize them as $x_2 = s$ and $x_4 = t$. The solution may be written as

$$\begin{aligned} x_1 &= 2s - t - 4 \\ x_2 &= s \\ x_3 &= 2t + 5 \\ x_4 &= t \\ x_5 &= 2. \end{aligned}$$

12. Show that a system of homogeneous linear equations is always consistent.

*Solution:*

A system of homogeneous linear equations has form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0. \end{aligned}$$

Then $x_1 = x_2 = \ldots = x_n = 0$ is automatically a solution: clearly

$$0a_{i1} + 0a_{i2} + \ldots + 0a_in = 0.$$