1. Use cofactor expansion to calculate the determinant of the matrix

$$A = \begin{pmatrix} 0 & 4 & 3 & 1 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix}.$$

   Include *all* of your work.

2. Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let B be the matrix obtained from A by multiplying row k of A by the constant c. Prove that $\det(B) = c \det(A)$.

3. Let A be an $n \times n$ matrix. Use induction to prove that for any constant c, $\det(cA) = c^n \det(A)$. **Note:** The determinant of a 1×1 matrix $A = (a_{11})$ is just $\det(A) = a_{11}$.

4. Prove that, for any square matrix A, $\det(A) = \det(A^\top)$.

5. Prove that, for any square matrix A, $\det(A^*) = \overline{\det(A)}$. You may use standard results on complex numbers without proof, as long as the results are stated clearly.

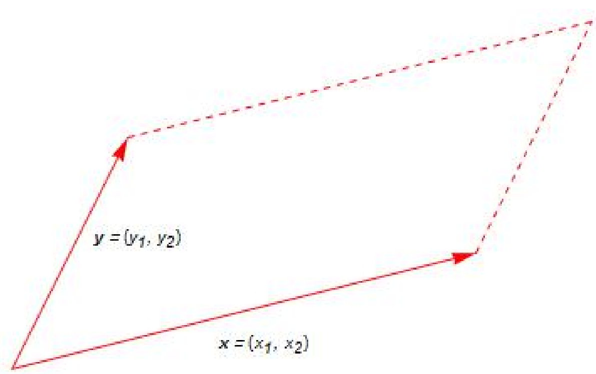
6. An $n \times n$ matrix A (with real or complex entries) is called *skew-symmetric* if $A^\top = -A$. Find a (nonzero) example of a 3×3 skew-symmetric matrix.

7. Let A be a skew-symmetric matrix.

   (a) Prove that if n is odd, then $\det(A) = 0$.

   (b) Find an example of an $n \times n$ skew-symmetric matrix A, n even, so that $\det(A) \neq 0$.

8. Let $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, where $x_i,\ y_i \in \mathbb{R}$, be nonparallel vectors in the xy plane written in component form (for example, x is the vector starting at the origin and ending at the point (x_1, x_2)). Consider the parallelogram below:



   Prove that this parallelogram has area

$$|\det \begin{pmatrix} x_1 & x_2 \\ y_2 & y_2 \end{pmatrix}|.$$

   (Hint: use vector projections and other standard calculus results; you may use these without proof, as long as you clearly state the results that you are using).*

9. A matrix A is called *orthogonal* if $A^\top = A^{-1}$.

   (a) Find an example of a 2×2 orthogonal matrix (other than I) with all real entries.

   (b) Prove that if A is orthogonal, then $\det(A) = \pm 1$.**

   (c) **Bonus:** Prove that a 2×2 real orthogonal matrix A with $\det(A) = 1$ has form
   $$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$
   where θ is any real number.***

10. Find an example of a system of 3 equations in 3 unknowns that has no solutions. Use geometric reasoning to explain why the system has no solution.

11. Use Gauss-Jordan elimination to solve the system
$$\begin{aligned} 6x_1 - 12x_2 + -5x_3 + 16x_4 - 2x_5 &= -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 &= 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 &= 33. \end{aligned}$$
    Include all of your work in your solution. If there is no solution, write “inconsistent.” If there are infinitely many solutions, paramaterize all free variables in the solution.

12. Show that a system of homogeneous linear equations is always consistent.

*This result actually extends to higher dimensions using a similar technique; for example, the volume of a parallelepiped defined by a triple of vectors in $\mathbb{R}^3$ can be calculated using the determinant of a 3×3 matrix whose rows are the vectors.

**This result, combined with the fact that the determinant is a continuous function, implies that the set (actually a Lie group) of all orthogonal matrices is disconnected in a topological sense; there is a connected “+1 determinant” piece and a connected “−1 determinant” piece.

***A matrix of this form is called a “rotation matrix.” We will see later in the course that such a matrix actually does act like a rotation on vectors.