1. Find a pair of 2×2 invertible matrices A and B so that $(AB)^{-1} \neq A^{-1}B^{-1}$.

   Example: Setting
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix},$$
   we see that
$$AB = \begin{pmatrix} 2 & 5 \\ 8 & 6 \end{pmatrix},$$
$$A^{-1} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix},\ B^{-1} = \frac{-1}{4}\begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix},\ \text{and } (AB)^{-1} = \frac{-1}{28}\begin{pmatrix} 6 & -5 \\ -8 & 2 \end{pmatrix}.$$
   However,
$$A^{-1}B^{-1} = \frac{-1}{28}\begin{pmatrix} 6 & -8 \\ -5 & 2 \end{pmatrix}$$
   which, comparing entries, we see is clearly not $(AB)^{-1}$.

2. Let A, B be $n \times n$. Prove that $(AB)^\top = B^\top A^\top$ (I am only asking you to prove for square matrices, but the theorem is true as long as the product AB is defined).

   Solution: Let a_{ij}, b_{ij}, p_{ij}, and q_{ij} be the entries of A, B, $(AB)^\top$, and $B^\top A^\top$ respectively. Now p_{ij} is the j,i entry of AB, so
$$p_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki}.$$
   On the other hand, the i,j entry of $B^\top A^\top$ is the scalar product of row i of $B^\top$ (that is, column i of B) and column j of $A^\top$ (that is, row j of A). Thus
$$\begin{aligned} q_{ij} &= \sum_{k=1}^{n} b_{ki}a_{jk} \\ &= \sum_{k=1}^{n} a_{jk}b_{ki} \\ &= p_{ij}. \end{aligned}$$
   Thus $(AB)^\top = B^\top A^\top$.

3. Find a pair of 2×2 symmetric matrices A and B so that AB *is not* symmetric.

   Example: Using
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$
   from the previous example, we see that
$$AB = \begin{pmatrix} 2 & 5 \\ 8 & 6 \end{pmatrix} \neq (AB)^\top$$
   is not symmetric.

4. Let A and B be symmetric matrices. Find a condition on A and B that is equivalent to the statement "AB is symmetric", and prove it. Your result should say "If A and B are symmetric, then AB is symmetric if and only if...".

   Solution: If A and B are symmetric, then AB is symmetric if and only if A and B commute.

   Proof: If AB is symmetric, then $AB = (AB)^\top$. However, using the identity proved above, we also know that $(AB)^\top = B^\top A^\top$. Combining $AB = B^\top A^\top$ with the fact that A and B are also symmetric, we see that $AB = BA$, that is A and B commute.

   On the other hand, if A and B commute, we have

$$\begin{aligned} (AB)^\top &= B^\top A^\top \\ &= BA \\ &= AB. \end{aligned}$$

   Thus AB is symmetric.

5. An $n \times n$ matrix A is called *skew hermitian* if $A^* = -A$. Find an example of a 3×3 skew hermitian matrix, *none* of whose entries is strictly real.

   Example: Set

$$A = \begin{pmatrix} 3i & 1+i & 2-i \\ -1+i & i & 3+2i \\ -2-i & -3+2i & 5i \end{pmatrix}.$$

   Then

$$\begin{aligned} A^* &= (\overline{A})^\top \\ &= \begin{pmatrix} -3i & 1-i & 2+i \\ -1-i & -i & 3-2i \\ -2+i & -3-2i & -5i \end{pmatrix}^\top \\ &= \begin{pmatrix} -3i & -1-i & -2+i \\ 1-i & -i & -3-2i \\ 2+i & 3-2i & -5i \end{pmatrix} \\ &= -A. \end{aligned}$$

6. Every $n \times n$ matrix A can be written in the form

$$A = A_S + A_H,$$

   where A_S is skew hermitian and A_H is hermitian. Find formulas for A_S and A_H. (Note: just as $(A^\top)^\top = A$, it is easy to see that $(A^*)^* = A$).

   Solution: Set

$$A_S = \frac{1}{2}(A - A^*) \text{ and } A_H = \frac{1}{2}(A + A^*).$$

Now it is easy to see that

$$(A+B)^* = A^* + B^*;$$

applying this fact to A_S and A_H, we see that

$$\begin{aligned} A_S^* &= \frac{1}{2}(A - A^*)^* \\ &= \frac{1}{2}(A^* - (A^*)^*) \\ &= \frac{1}{2}(A^* - A) \\ &= -\frac{1}{2}(A - A^*) \\ &= -A_S, \end{aligned}$$

and

$$\begin{aligned} A_H^* &= \frac{1}{2}(A + A^*)^* \\ &= \frac{1}{2}(A^* + (A^*)^*) \\ &= \frac{1}{2}(A^* + A) \\ &= A_H. \end{aligned}$$

Thus A_S is skew hermitian, A_H is hermitian, and

$$\begin{aligned} A_S + A_H &= \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*) \\ &= \frac{1}{2}A - \frac{1}{2}A^* + \frac{1}{2}A + \frac{1}{2}A^* \\ &= \frac{1}{2}A + \frac{1}{2}A - \frac{1}{2}A^* + \frac{1}{2}A^* \\ &= A. \end{aligned}$$

7. Let A be an $n \times n$ matrix with strictly real entries. If A is also skew hermitian, what can we say about the diagonal entries of A?

   Solution: $A^* = -A$ implies that $\overline{a}_{ii} = -a_{ii}$. However, since a_{ii} is a real number, we also have $\overline{a}_{ii} = a_{ii}$. Thus $a_{ii} = -a_{ii}$ implies that all of the diagonal entries of A are 0s.

8. Let A be an $n \times n$ matrix, and r a positive integer. We define powers of A in a natural way:

$$A^1 = A,\ A^2 = A \cdot A,\ \ldots,\ A^r = \underbrace{A \cdot A \cdot \ldots \cdot A}_{r \text{ factors}}.$$

   Then it is clear that the usual exponential rules hold, i.e.

$$A^r A^s = A^{r+s} \text{ and } (A^r)^s = A^{rs}.$$

Use induction to prove that, if A is invertible, r a positive integer, then A^r is invertible as well, and

$$(A^r)^{-1} = (A^{-1})^r.$$

Solution: If $r = 1$, $A^1 = A$, then

$$(A^1)^{-1} = A^{-1} = (A^{-1})^1.$$

Assume the statement holds for $r = n$, that is

$$(A^n)^{-1} = (A^{-1})^n,$$

and consider the product

$$\begin{aligned}(A^{n+1})(A^{-1})^{n+1} &= (AA^n)((A^{-1})^n A^{-1}) \\ &= A(A^n(A^{-1})^n)A^{-1} \\ &= AA^{-1} \\ &= I,\end{aligned}$$

so $(A^{-1})^{n+1}$ is the inverse of A^{n+1}.

9. Prove that if $A = [a_{ij}]$ is an $n \times n$ upper triangular matrix, r a positive integer, then the diagonal entries of A^r have form a_{ii}^r.

   Solution: We proceed by induction: the statement is clearly true for $r = 1$, that is the diagonal entries of $A^1 = A$ have form $a_{ii}^1 = a_{ii}$.

   Assume the statement is true for $r = n$. Since the product of a pair of upper triangular matrices is also upper triangular, the ith row of A^n has form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_{ii}^n & u_{i,i+1} & \dots & u_{in} \end{pmatrix}.$$

   Now the i, i entry of A^{n+1} is the scalar product of row i of A^n and column i of A; column i has form

$$\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

   Now the first $i-1$ entries of the ith row of A^n are all 0s, as are the last $n-i$ entries of column i; that is, the scalar product of row i of A^n and column i of A has form

$$0 \cdot a_{1i} + 0 \cdot a_{2i} + \ldots + 0 \cdot a_{i-1,i} + a_{ii}^n \cdot a_{ii} + 0 \cdot u_{i,i+1} + \ldots + 0 \cdot u_{in} = a_{ii}^{n+1}.$$

   Thus the diagonal entries have the desired form.