1. Find a pair of $2 \times 2$ invertible matrices $A$ and $B$ so that $(AB)^{-1} \neq A^{-1}B^{-1}$.

*Example:* Setting 

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix},$$

we see that 

$$AB = \begin{pmatrix} 2 & 5 \\ 8 & 6 \end{pmatrix},$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}, \quad B^{-1} = \frac{-1}{4} \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}, \quad \text{and} \quad (AB)^{-1} = \frac{-1}{28} \begin{pmatrix} 6 & -5 \\ -8 & 2 \end{pmatrix}.$$

However, 

$$A^{-1}B^{-1} = \frac{-1}{28} \begin{pmatrix} 6 & -8 \\ -5 & 2 \end{pmatrix}$$

which, comparing entries, we see is clearly not $(AB)^{-1}$.

2. Let $A, B$ be $n \times n$. Prove that $(AB)^\top = B^\top A^\top$ (I am only asking you to prove for square matrices, but the theorem is true as long as the product $AB$ is defined).

*Solution:* Let $a_{ij}, b_{ij}, p_{ij},$ and $q_{ij}$ be the entries of $A$, $B$, $(AB)^\top$, and $B^\top A^\top$ respectively. Now $p_{ij}$ is the $j, i$ entry of $AB$, so

$$p_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki}.$$ 

On the other hand, the $i, j$ entry of $B^\top A^\top$ is the scalar product of row $i$ of $B^\top$ (that is, column $i$ of $B$) and column $j$ of $A^\top$ (that is, row $j$ of $A$). Thus

$$q_{ij} = \sum_{k=1}^{n} b_{ki}a_{jk}$$

$$= \sum_{k=1}^{n} a_{jk}b_{ki}$$

$$= p_{ij}.$$ 

Thus $(AB)^\top = B^\top A^\top$.

3. Find a pair of $2 \times 2$ symmetric matrices $A$ and $B$ so that $AB$ is not symmetric.

*Example:* Using 

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

from the previous example, we see that 

$$AB = \begin{pmatrix} 2 & 5 \\ 8 & 6 \end{pmatrix} \neq (AB)^\top$$

is not symmetric.
4. Let \( A \) and \( B \) be symmetric matrices. Find a condition on \( A \) and \( B \) that is equivalent to the statement “\( AB \) is symmetric”, and prove it. Your result should say “If \( A \) and \( B \) are symmetric, then \( AB \) is symmetric if and only if...”.

**Solution:** If \( A \) and \( B \) are symmetric, then \( AB \) is symmetric if and only if \( A \) and \( B \) commute.

**Proof:** If \( AB \) is symmetric, then \( AB = (AB)^\top \). However, using the identity proved above, we also know that \( (AB)^\top = B^\top A^\top \). Combining \( AB = B^\top A^\top \) with the fact that \( A \) and \( B \) are also symmetric, we see that \( AB = BA \), that is \( A \) and \( B \) commute.

On the other hand, if \( A \) and \( B \) commute, we have
\[
(AB)^\top = B^\top A^\top = BA = AB.
\]
Thus \( AB \) is symmetric.

5. An \( n \times n \) matrix \( A \) is called *skew hermitian* if \( A^* = -A \). Find an example of a \( 3 \times 3 \) skew hermitian matrix, *none* of whose entries is strictly real.

**Example:** Set
\[
A = \begin{pmatrix}
3i & 1+i & 2-i \\
-1+i & i & 3+2i \\
-2-i & -3+2i & 5i
\end{pmatrix}.
\]

Then
\[
A^* = (\overline{A})^\top = \begin{pmatrix}
-3i & 1-i & 2+i \\
-1-i & -i & 3-2i \\
-2+i & -3-2i & -5i
\end{pmatrix}^\top
= \begin{pmatrix}
-3i & -1-i & -2+i \\
1-i & -i & -3-2i \\
2+i & 3-2i & -5i
\end{pmatrix}
= -A.
\]

6. Every \( n \times n \) matrix \( A \) can be written in the form
\[
A = A_S + A_H,
\]
where \( A_S \) is skew hermitian and \( A_H \) is hermitian. Find formulas for \( A_S \) and \( A_H \). (Note: just as \( (A^\top)^\top = A \), it is easy to see that \( (A^*)^* = A \)).

**Solution:** Set
\[
A_S = \frac{1}{2}(A - A^*) \quad \text{and} \quad A_H = \frac{1}{2}(A + A^*).
\]
Now it is easy to see that 
\[(A + B)^* = A^* + B^*;\]
applying this fact to \(A_S\) and \(A_H\), we see that
\[
A_S^* = \frac{1}{2}(A - A^*)^*
= \frac{1}{2}(A^* - (A^*)^*)
= \frac{1}{2}(A^* - A)
= -\frac{1}{2}(A - A^*)
= -A_S,
\]
and
\[
A_H^* = \frac{1}{2}(A + A^*)^*
= \frac{1}{2}(A^* + (A^*)^*)
= \frac{1}{2}(A^* + A)
= A_H.
\]

Thus \(A_S\) is skew hermitian, \(A_H\) is hermitian, and
\[
A_S + A_H = \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*)
= \frac{1}{2}A - \frac{1}{2}A^* + \frac{1}{2}A + \frac{1}{2}A^*
= \frac{1}{2}A \leq \frac{1}{2}A^* + \frac{1}{2}A^*
= A.
\]

7. Let \(A\) be an \(n \times n\) matrix with strictly real entries. If \(A\) is also skew hermitian, what can we say about the diagonal entries of \(A\)?

Solution: \(A^* = -A\) implies that \(\overline{a_{ii}} = -a_{ii}\). However, since \(a_{ii}\) is a real number, we also have \(a_{ii} = a_{ii}\). Thus \(a_{ii} = -a_{ii}\) implies that all of the diagonal entries of \(A\) are 0s.

8. Let \(A\) be an \(n \times n\) matrix, and \(r\) a positive integer. We define powers of \(A\) in a natural way:
\[
A^1 = A, \ A^2 = A \cdot A, \ldots, \ A^r = A \cdot A \cdot \ldots \cdot A.
\]

Then it is clear that the usual exponential rules hold, i.e.
\[
A^r A^s = A^{r+s} \text{ and } (A^r)^s = A^{rs}.
\]
Use induction to prove that, if $A$ is invertible, $r$ a positive integer, then $A^r$ is invertible as well, and

$$(A^r)^{-1} = (A^{-1})^r.$$ 

**Solution:** If $r = 1$, $A^1 = A$, then

$$(A^1)^{-1} = A^{-1} = (A^{-1})^1.$$ 

Assume the statement holds for $r = n$, that is

$$(A^n)^{-1} = (A^{-1})^n,$$

and consider the product

$$(A^{n+1})(A^{-1})^{n+1} = (AA^n)((A^{-1})^nA^{-1})$$

$$= A(A^n(A^{-1})^nA^{-1}$$

$$= AA^{-1}$$

$$= I,$$

so $(A^{-1})^{n+1}$ is the inverse of $A^{n+1}$.

9. Prove that if $A = [a_{ij}]$ is an $n \times n$ upper triangular matrix, $r$ a positive integer, then the diagonal entries of $A^r$ have form $a_{rr}^r$.

**Solution:** We proceed by induction: the statement is clearly true for $r = 1$, that is the diagonal entries of $A^1 = A$ have form $a_{ii}^1 = a_{ii}$.

Assume the statement is true for $r = n$. Since the product of a pair of upper triangular matrices is also upper triangular, the $i$th row of $A^n$ has form

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & a_{ii}^n & u_{i,i+1} & \ldots & u_{in}
\end{pmatrix}.
$$

Now the $i, i$ entry of $A^{n+1}$ is the scalar product of row $i$ of $A^n$ and column $i$ of $A$; column $i$ has form

$$
\begin{pmatrix}
a_{1i} \\
a_{2i} \\
\vdots \\
a_{ii} \\
0 \\
\vdots \\
0
\end{pmatrix}.
$$

Now the first $i - 1$ entries of the $i$th row of $A^n$ are all 0s, as are the last $n - i$ entries of column $i$; that is, the scalar product of row $i$ of $A^n$ and column $i$ of $A$ has form

$$0 \cdot a_{1i} + 0 \cdot a_{2i} + \ldots + 0 \cdot a_{i-1,i} + a_{ii}^n \cdot a_{ii} + 0 \cdot u_{i,i+1} + \ldots + 0 \cdot u_{in} = a_{ii}^{n+1}.$$ 

Thus the diagonal entries have the desired form.