1. Every complex number has form $a+bi$, where $a$ and $b$ are *real* numbers. In addition, every complex number other than 0 has a multiplicative inverse, i.e., another complex number $c+di$ so that $(a+bi)(c+di)=1$. Suppose that $a$ and $b$ are real numbers, and at least one of them is not 0. Find the real numbers $c$ and $d$ so that $c+di$ is the multiplicative inverse of $a+bi$. Hint: start by thinking of $c+di$ as the number

   $$c+di=\frac{1}{a+bi}.$$

   Then use a calculus trick to rewrite this fraction with no complex numbers in the denominator.

   *Solution:* We can think of the inverse as

   $$\begin{aligned} c+di &= \frac{1}{a+bi} \\ &= \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2-b^2} \\ &= \frac{a}{a^2-b^2}+\frac{-b}{a^2-b^2}i. \end{aligned}$$

   So

   $$c=\frac{a}{a^2-b^2} \text{ and } d=\frac{-b}{a^2-b^2}.$$

2. Find two distinct square roots of $i$.

   *Solution:* Suppose that $a+bi$ is a number so that $i=(a+bi)^2$. Then

   $$i = a^2-b^2+2abi,$$

   so that

   $$a^2-b^2=0 \text{ and } 2ab=1.$$

   The first equality implies that $a=\pm b$. If $a=b$, then $2a^2=1$, so $a=b=\pm 1/\sqrt{2}$. So

   $$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i \text{ and } \frac{-1}{\sqrt{2}}+\frac{-1}{\sqrt{2}}i$$

   are both roots of $i$. We do not have to consider the case $a=-b$, because it will produce the same results.

3. Matrices $A$, $B$ and $C$ are given by

   $$A=\begin{pmatrix}1 & 1 & -3\\ 2 & 1 & -4\end{pmatrix},\ B=\begin{pmatrix}4 & 1 & 0\\ -1 & 3 & 1\\ 0 & -5 & 1\end{pmatrix},\ \text{and } C=\begin{pmatrix}0 & 0 & -1\\ 4 & 2 & 0\\ -1 & 3 & 1\end{pmatrix}.$$

(a) Find $A^\top$.

*Solution:*

$$A^\top = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ -3 & -4 \end{pmatrix}.$$

(b) Calculate $AB$.

*Solution:*

$$AB = \begin{pmatrix} 3 & 19 & -2 \\ 7 & 25 & -3 \end{pmatrix}.$$

(c) Calculate $B - C$.

$$B - C = \begin{pmatrix} 4 & 1 & 1 \\ -5 & 1 & 1 \\ 1 & -8 & 0 \end{pmatrix}.$$

(d) Find the matrix $3C$.

*Solution:*

$$3C = \begin{pmatrix} 0 & 0 & -3 \\ 12 & 6 & 0 \\ -3 & 9 & 3 \end{pmatrix}.$$

4. Matrix $A$ has size $3 \times 5$; $B$ and $C$ are both $5 \times 7$; and $D$ has size $3 \times 3$. If the following calculations are possible, determine the size of the resulting matrix. Otherwise, write "not defined".

   (a) $AB$
   *Solution:* Size $3 \times 7$

   (b) $BA$
   *Solution:* Not defined.

   (c) $B + C$
   *Solution:* Size $5 \times 7$

   (d) $A^\top D$ *Solution:* Size $5 \times 3$

5. Suppose that $A$ is an $m \times n$ matrix, and that $B$ is a matrix so that $AB$ and $BA$ are both defined. Prove that $B$ has size $n \times m$.

   *Solution:* Suppose that $B$ is an $r \times s$ matrix. $AB$ is defined, so $r = n$, and $BA$ is defined, so $s = m$. Thus $B$ is $n \times m$.