Challenge Problem 3 Key

1. A skew-symmetric matrix $Y$ has the property that $Y^\top = -Y$; so any $2 \times 2$ skew-symmetric matrix may be written in the form

$$Y = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Setting

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have $Y = \theta X$ as desired; again, any $2 \times 2$ skew-symmetric matrix may be written in this form.

2. In challenge problem 1, you actually calculated $\exp(\theta X)$; indeed,

$$\exp(\theta X) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

3. The bonus problem on Homework 3 was to show that any real $2 \times 2$ determinant 1 orthogonal matrix can be written in the form

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. Thus the image of $\mathfrak{o}(2, \mathbb{R})$ under $\exp$ is precisely $\text{SO}(2, \mathbb{R})$. The map

$$\exp : \mathfrak{o}(2, \mathbb{R}) \mapsto \text{SO}(2, \mathbb{R})$$

is onto, but definitely not one-to-one.