Recall that the *exponential of a matrix* is defined using the Taylor series expansion of $e^x$: in particular, for any $n \times n$ matrix $A$,

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \dots,$$

where $A^0 = I$. The exponential $\exp(A)$ converges to an $n \times n$ matrix $a$.

In this challenge problem, we will consider two different sets of matrices, and their interconnections via the exponential. We are interested in:

$$\begin{aligned} \mathfrak{o}(2, \mathbb{R}) &= \left\{ Y \in \mathcal{M}_2(\mathbb{R}) | Y^\top = -Y \right\} \text{ and} \\ \mathrm{SO}(2, \mathbb{R}) &= \left\{ y \in \mathcal{M}_2(\mathbb{R}) \middle| y^\top = y^{-1}, \ \det y = 1 \right\}. \end{aligned}$$

In particular, $\mathfrak{o}(2, \mathbb{R})$ is the set real of $2 \times 2$ skew-symmetric matrices, and $\mathrm{SO}(2, \mathbb{R})$ is the set of real $2 \times 2$ determinant 1 orthogonal matrices.

In this challenge problem, we will prove that the image of $\mathfrak{o}(2, \mathbb{R})$ under exp is precisely $\mathrm{SO}(2, \mathbb{R})$.

1. Find the form of any $2 \times 2$ matrix $Y$ in $\mathfrak{o}(2, \mathbb{R})$; you should write your matrix as $Y = \theta X$, where $\theta$ is a real variable and $X$ is a *specific* matrix in $\mathfrak{o}(2, \mathbb{R})$.

2. Calculate $\exp(\theta X)$. (*Hint*: you have already made this calculation in a previous challenge problem!)

3. Show that *every* matrix in $\mathrm{SO}(2, \mathbb{R})$ is of the form $\exp(\theta X)$ for your choice of $X$. (*Hint*: this problem showed up on Homework 3!)