1. Calculate
$$\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta).$$

   Note that, since the exponential function converges in norm, we may calculate the derivative term-by-term from the Taylor series for $\exp(\theta X)$.

   Solution: Since $\exp(\theta X)$ has (converging) Taylor Series
$$\exp(\theta X) = I + \theta X + \frac{1}{2!}(\theta X)^2 + \frac{1}{3!}(\theta X)^3 + \frac{1}{4!}(\theta X)^4 + \ldots,$$
   we calculate the derivative term-by-term from the series:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\exp(\theta X) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\big(I + \theta X + \frac{1}{2!}(\theta X)^2 + \frac{1}{3!}(\theta X)^3 + \frac{1}{4!}(\theta X)^4 + \ldots\big) \\ &= 0 + X + \theta X^2 + \frac{1}{2!}\theta^2 X^3 + \frac{1}{3!}\theta^3 X^4 + \ldots \\ &= X\big(I + \theta X + \frac{1}{2!}(\theta X)^2 + \frac{1}{3!}(\theta X)^3 + \frac{1}{4!}(\theta X)^4 + \ldots\big) \\ &= X\exp(\theta X).\end{aligned}$$

   This formula should look familiar to you from calculus–indeed, for any constant (real or complex) x and real or complex variable θ, we know via the chain rule that
$$\frac{\mathrm{d}}{\mathrm{d}\theta} e^{x\theta} = x e^{x\theta}.$$

2. Show that $\exp(-\theta X) = (\exp(\theta X))^{-1}$.

   Solution: We have already calculate the matrix $\exp(\theta X)$ as
$$\exp(\theta X) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$
   using the formula for inverses, we see that
$$\begin{aligned}\exp(\theta X)^{-1} &= \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.\end{aligned}$$

   Now let's calculate $\exp(-\theta X)$ using the Taylor series expansion:
$$\exp(-\theta X) = I - \theta X + \frac{1}{2!}(\theta X)^2 - \frac{1}{3!}(\theta X)^3 + \frac{1}{4!}(\theta X)^4 - \ldots;$$

notice that the only terms that change are the odd ones. We calculated the powers of θX in the last challenge problem; using these results, we see that

$$\begin{aligned}
-\theta X &= \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \\
\theta^2 X^2 &= \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} \\
-\theta^3 X^3 &= \begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix} \\
\theta^4 X^4 &= \begin{pmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{pmatrix} \\
-\theta^5 X^5 &= \begin{pmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{pmatrix},
\end{aligned}$$

etc. Thus the $1,1$ and $2,2$ entries of $\exp(-\theta X)$ are identical to the corresponding entries of $\exp(\theta X)$, while the $1,2$ and $2,1$ entries change signs. Thus we see that

$$\exp(-\theta X) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \exp(\theta X)^{-1}.$$

Again, this should remind you of an identity that you are familiar with: for any real constant x and real or complex variable θ,

$$(e^{\theta x})^{-1} = e^{-\theta x}.$$

What's more, this rule is true in general: given any $n \times n$ matrix X, the $n \times n$ matrix $\exp(X)$ is invertible, and

$$\exp(-X) = \exp(X)^{-1}.$$

3. Let a be *any* invertible 2×2 matrix. Use the Taylor series expansion of $\exp(\theta X)$ to show that

$$a(\exp(X))a^{-1} = \exp(a(X)a^{-1}).$$

Solution: Starting with the Taylor series expansion

$$\exp X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \dots$$

and adjusting the input for exp to aXa^{-1}, we have

$$\begin{aligned}
\exp(aXa^{-1}) &= I + aXa^{-1} + \frac{1}{2!}(aXa^{-1})^2 + \frac{1}{3!}(aXa^{-1})^3 + \frac{1}{4!}(aXa^{-1})^4 + \dots \\
&= I + aXa^{-1} + \frac{1}{2!}(aXa^{-1}aXa^{-1}) + \frac{1}{3!}(aXa^{-1}aXa^{-1}aXa^{-1}) \\
&\quad + \frac{1}{4!}(aXa^{-1}aXa^{-1}aXa^{-1}aXa^{-1})^4 + \dots \\
&= I + aXa^{-1} + \frac{1}{2!}(aXXa^{-1}) + \frac{1}{3!}(aXXXa^{-1}) + \frac{1}{4!}(aXXXXa^{-1}) + \dots \\
&= I + aXa^{-1} + \frac{1}{2!}(aX^2a^{-1}) + \frac{1}{3!}(aX^3a^{-1}) + \frac{1}{4!}(aX^4a^{-1}) + \dots \\
&= a(\exp(X))a^{-1},
\end{aligned}$$

since the series converges.