In the last challenge problem, we defined the *exponential of a matrix* using the Taylor series expansion of e^x: in particular, for any $n \times n$ matrix A,

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \dots,$$

where $A^0 = I$. The exponential $\exp(A)$ converges to an $n \times n$ matrix a.

We considered the matrix

$$A(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where θ is a real or complex variable. Set

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

we think of $A(\theta) = \theta X$.

We found that

$$\exp(A(\theta)) = \exp(\theta X) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The function $f : \mathbb{C} \to \mathcal{M}_n$ defined by

$$f(\theta) = \exp(\theta X)$$

is called the *one-parameter subgroup generated by X*.

1. Calculate
$$\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta).$$
Note that, since the exponential function converges in norm, we may calculate the derivative term-by-term from the Taylor series for $\exp(\theta X)$.

2. Show that $\exp(-\theta X) = (\exp(\theta X))^{-1}$.

3. Let a be *any* invertible 2×2 matrix. Use the Taylor series expansion of $\exp(\theta X)$ to show that
$$a(\exp(X))a^{-1} = \exp(a(X)a^{-1}).$$