Solutions:

1. Since $A = 0$ and $0^k = 0$ for all integers $k > 0$, we have

$$\exp(A) = I + 0 + 0 + \ldots = I.$$ 

2. Let’s compute powers of $A(\theta)$:

$$A(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

$$A(\theta)^2 = \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix}$$

$$A(\theta)^3 = \begin{pmatrix} 0 & -\theta^3 \\ 0 & 0 \end{pmatrix}$$

$$A(\theta)^4 = \begin{pmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{pmatrix}$$

$$A(\theta)^5 = \begin{pmatrix} 0 & \theta^5 \\ -\theta^5 & 0 \end{pmatrix}$$

You can see at this point that the pattern repeats; thus we are ready to compute the entries of

$$a(\theta) = \exp(A(\theta)) = \sum_{i=0}^{\infty} \frac{A(\theta)^n}{n!} = I + A(\theta) + \frac{A(\theta)^2}{2!} + \ldots$$

Let $a_{ij}$ denote the entries of $a(\theta)$. We see from above that

$$a_{11} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$= \cos \theta;$$

$$a_{12} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$= \sin \theta;$$

$$a_{21} = -a_{12}$$

$$= -\sin \theta;$$

and

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\[ a_{22} = a_{11} = \cos \theta. \]

Thus the matrix \( a(\theta) = \exp(A(\theta)) \) has form

\[
a(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

3. It is easy to see that \( \text{tr}(A) = 0 \), and \( \text{tr}(A(\theta)) = 0 \) for all \( \theta \). On the other hand, \( \det(\exp(A)) = 1 \) and \( \det(\exp(A(\theta))) = 1 \) for all \( \theta \).

4. The behavior observed above is no fluke–for any trace 0 matrix \( A \), its exponential \( \exp(A) \) is a determinant 1 matrix. More generally, it is known that

\[
\det(\exp(A)) = e^{\text{tr}(A)}.
\]