Solutions:

1. Since $A = \mathbf{0}$ and $\mathbf{0}^k = \mathbf{0}$ for all integers $k > 0$, we have

$$\exp(A) = I + \mathbf{0} + \mathbf{0} + \ldots = I.$$

2. Let's compute powers of $A(\theta)$:

$$\begin{aligned}
A(\theta) &= \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \\
A(\theta)^2 &= \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} \\
A(\theta)^3 &= \begin{pmatrix} 0 & -\theta^3 \\ \theta^3 & 0 \end{pmatrix} \\
A(\theta)^4 &= \begin{pmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{pmatrix} \\
A(\theta)^5 &= \begin{pmatrix} 0 & \theta^5 \\ -\theta^5 & 0 \end{pmatrix}
\end{aligned}$$

You can see at this point that the pattern repeats; thus we are ready to compute the entries of

$$a(\theta) = \exp(A(\theta)) = \sum_{i=0}^{\infty} \frac{A(\theta)^n}{n!} = I + A(\theta) + \frac{A(\theta)^2}{2!} + \ldots$$

Let a_{ij} denote the entries of $a(\theta)$. We see from above that

$$\begin{aligned}
a_{11} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \\
&= \cos\theta;
\end{aligned}$$

$$\begin{aligned}
a_{12} &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \\
&= \sin\theta;
\end{aligned}$$

$$\begin{aligned}
a_{21} &= -a_{12} \\
&= -\sin\theta;
\end{aligned}$$

and

$$\begin{aligned} a_{22} &= a_{11} \\ &= \cos\theta. \end{aligned}$$

Thus the matrix $a(\theta) = \exp(A(\theta))$ has form

$$a(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

3. It is easy to see that $\operatorname{tr}(A) = 0$, and $\operatorname{tr}(A(\theta)) = 0$ for all θ. On the other hand, $\det(\exp(A)) = 1$ and $\det(\exp(A(\theta))) = 1$ for all θ.

4. The behavior observed above is no fluke–for any trace 0 matrix A, its exponential $\exp(A)$ is a determinant 1 matrix. More generally, it is known that

$$\det(\exp(A)) = e^{\operatorname{tr}(A)}.$$