Gauss-Jordan Elimination

Our main focus in this section is on a detailed discussion of a method for solving systems of equations. In the last section, we saw that the general procedure for solving a system is to:

1. Write the system's augmented equation.
2. Use elementary row operations to rewrite the augmented matrix in a simpler form (i.e., one whose solutions are easy to find).
3. Since elementary row operations *do not alter solutions*, the solutions found above are also solutions to the original system.

The hard part of the process is step 2–using elementary row operations to write the augmented matrix in a simpler form. What form should we use, and how do we choose row operations appropriately to get to that form?

To get some motivation for the methods in this section, let's take another look at the example from the previous section: we saw that the system

$$\begin{aligned} x_3 &= \tfrac{1}{2} \\ x_1 + x_2 &= 7 \\ x_1 &= 4, \end{aligned}$$

has augmented matrix

$$\begin{pmatrix} 0 & 0 & 1 & | & \frac{1}{2} \\ 1 & 1 & 0 & | & 7 \\ 1 & 0 & 0 & | & 4 \end{pmatrix}.$$

After applying elementary row operations to this matrix, we ended up with the augmented matrix

$$\begin{pmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix},$$

whose solution is the same as the solution to the original system. The form of this last augmented matrix makes it quite clear that

$$(4, 3, \tfrac{1}{2})$$

is a solution.

In general, the form we want to create in the augmented matrix is that of the last matrix in the example above because of the ease involved in finding its solutions. Let's investigate the form in more detail:

$$\begin{pmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

is simple to solve since each row corresponds to an equation with *only one* unknown. For example, row 1

$$\begin{pmatrix} 1 & 0 & 0 & 4 \end{pmatrix}$$

of the augmented matrix indicates that the coefficients of x_2 and x_3 are both 0, thus leaving no ambiguity as to the value of x_1. The other rows of the matrix are similar.

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

is an example of a matrix in *reduced row echelon form*, which is the form we'll want for finding solutions. Let's look at the properties a matrix must have to be in reduced row echelon form:

Definition. A matrix is in *row echelon form* if it satisfies properties $1-3$ below, and in *reduced row echelon form* if it satisfies all of the following properties:

1. Any row that does not consist entirely of 0s has a 1 as its first entry (called a *leading* 1).
2. Any rows that consist entirely of 0s appear at the bottom of the matrix.
3. If successive rows have nonzero entries, then the leading 1 of the uppermost row occurs in an entry *before* the leading 1 of the lower row.
4. Any column containing a leading 1 has 0s for all of its other entries.

Above, I claimed that

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

is in reduced row echelon form. Let's check the four properties above to be sure that this is true.

We'll start with property 1: *Any row that does not consist entirely of* 0*s has a* 1 *as its first nonzero entry.* Investigating the matrix, we notice that each row does indeed have this property. For example, the second row $\begin{pmatrix} 0 & 1 & 0 & 3 \end{pmatrix}$ has a 1 as its first nonzero entry, in the second position. The leading 1s of the matrix are highlighted below:

$$\begin{pmatrix} \color{red}{1} & 0 & 0 & 4 \\ 0 & \color{red}{1} & 0 & 3 \\ 0 & 0 & \color{red}{1} & \frac{1}{2} \end{pmatrix}.$$

Next, let's check property 2: *Any rows that consist entirely of* 0*s appear at the bottom of the matrix.* Of course, this property is easy to check for our matrix–there are no rows consisting entirely of 0s, so nothing to worry about.

Property 3 is a bit more complicated to check: *If successive rows have nonzero entries, then the leading* 1 *of the uppermost row occurs in an entry* before *the leading* 1 *of the lower row.* All 3 of our rows have nonzero entries, so we'll need to check all 3 for compliance. Let's start with the first two rows, whose leading 1s are highlighted:

$$\begin{pmatrix} \color{red}{1} & 0 & 0 & 4 \\ 0 & \color{red}{1} & 0 & 3 \end{pmatrix}.$$

Notice that the leading 1 of the first row occurs in the first position, while the leading 1 of the second row occurs later, in the second position; thus the first two rows check out.

Similarly, the second two rows check out:

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

The leading 1 in row 2 does indeed occur before the leading 1 in row 3.

At this point, we should note that our matrix satisfies properties $1-3$ of the definition, and is thus in row echelon form. To determine if it is in *reduced* row echelon form, let's check property 4: *Any column containing a leading* 1 *has* 0*s for all of its other entries.*

We have seen that the matrix has 3 leading 1s, so we'll need to check the three corresponding columns. The first leading 1 is highlighted below in red, and the rest of its column is highlighted in blue:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Notice that, with the exception of the leading 1, all the entries of column 1 are 0s, so that this column checks out.

The next leading 1 occurs in the second row; it is highlighted in red, and the rest of its column in blue:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Again, this row checks out–the only nonzero entry is the leading 1.

The final leading 1 occurs in the third row; again, the leading 1 is highlighted in red, and the rest of its column in blue:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

It is clear that this column checks out as well; thus our matrix is indeed in reduced row echelon form.

Example. The matrices below are all in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example. The matrices below are in row echelon form, but *not* in reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Determine why they fail to be in reduced row echelon form.

1. A matrix that is in row echelon form but *not* in reduced row echelon form must fail property 4: *Any column containing a leading* 1 *has* 0*s for all of its other entries.*

   So the matrix

   $$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

   must have a conflict with its leading 1s, which are highlighted below:

   $$\begin{pmatrix} \color{red}{1} & 2 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 3 \\ 0 & 0 & \color{red}{1} & 2 \end{pmatrix}.$$

   We need to check the columns that correspond to leading 1s. The first column appears to be ok–all of the entries other than the leading 1 are 0s:

   $$\begin{pmatrix} \color{red}{1} & 2 & 0 & 0 \\ \color{blue}{0} & 1 & 0 & 3 \\ \color{blue}{0} & 0 & 1 & 2 \end{pmatrix}.$$

   Let's look at the second leading 1, which occurs in the second column:

   $$\begin{pmatrix} 1 & \color{blue}{2} & 0 & 0 \\ 0 & \color{red}{1} & 0 & 3 \\ 0 & \color{blue}{0} & 1 & 2 \end{pmatrix}.$$

   Notice that, other than the leading 1, there *is* a nonzero entry in this column, the 2 highlighted in blue below:

   $$\begin{pmatrix} 1 & \color{blue}{2} & 0 & 0 \\ 0 & \color{red}{1} & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

   This 2 prevents the matrix from being in reduced row echelon form.

2. The matrix

   $$\begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has its leading 1s highlighted in red below:

$$\begin{pmatrix} \color{red}{1} & -3 & 0 & 1 & 0 & 2 \\ 0 & 0 & \color{red}{1} & 2 & 0 & 5 \\ 0 & 0 & 0 & \color{red}{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With the leading 1s highlighted, it becomes clear where the problem lies–in column 4, highlighted below:

$$\begin{pmatrix} 1 & -3 & 0 & \color{blue}{1} & 0 & 2 \\ 0 & 0 & 1 & \color{blue}{2} & 0 & 5 \\ 0 & 0 & 0 & \color{red}{1} & 0 & 3 \\ 0 & 0 & 0 & \color{blue}{0} & 0 & 0 \end{pmatrix}.$$

Since column 4 has a leading 1 *and* several nonzero entries, the matrix is not in reduced row echelon form.

Example. Find a reason that each of the matrices below fails to be in row echelon form:

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

1. The first matrix

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

fails to be in row echelon form because of property 1: *Any row that does not consist entirely of* 0*s has a* 1 *as its first entry*. The second row causes problems because its first nonzero entry is a 2:

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ \color{red}{0} & \color{red}{0} & \color{red}{2} & \color{red}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

is not in row echelon form because it fails property 2: *Any rows that consist entirely of* 0*s appear at the bottom of the matrix.* Notice that the second row, highlighted in red, consists entirely of 0 entries, but occurs above a row with nonzero entries:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

3. The matrix
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$
fails property 3: *If successive rows have nonzero entries, then the leading* 1 *of the uppermost row occurs in an entry* before *the leading* 1 *of the lower row.* The leading 1s of the matrix are highlighted below:
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$
The last two rows are causing a problem–notice that the leading 1 in row 3 occurs *before* the leading 1 in row 2:
$$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Reduced Row Echelon Form and Solution Types

In the previous section, we saw that the solution set of a system of equations must come in one of three forms:

1. A unique solution
2. Infinitely many solutions
3. No solution.

One reason that we prefer to write the augmented matrix for a system of equations in reduced row echelon form is that this form allows us to easily find both the solution set *and* the form of the solution, as indicated by the following example.

Example. Solve the systems of equations whose augmented matrices are written below in reduced row echelon form, and determine the form of the solution:
$$\begin{pmatrix} 1 & 0 & 3 & | & 2 \\ 0 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 0 & 0 & | & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & \frac{1}{2} \\ 0 & 0 & 1 & | & -3 \end{pmatrix}.$$

1. The augmented matrix
$$\begin{pmatrix} 1 & 0 & 3 & | & 2 \\ 0 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

corresponds to a system of three equations in three unknowns, say x, y, and z. The associated equations are

$$\begin{aligned} x + 3z &= 2 \\ y + 2z &= -1 \\ 0 &= 0, \end{aligned}$$

which reduces to

$$\begin{aligned} x + 3z &= 2 \\ y + 2z &= -1. \end{aligned}$$

In this example, the variables x and y corresponded to leading 1s in the augmented matrix, and are called *leading variables*. However, z did not correspond to a leading 1; because of this, there are no constraints on the choice of z. Since we are free to choose z in any way we wish, we refer to it as a *free variable*; in general, any variable that does not correspond to a leading 1 is called *free*.

We can write the solution to the system as

$$x = 2 - 3z, \qquad y = -1 - 2z.$$

However, it is standard practice to rewrite solutions involving a free variable via parameterization: we introduce a new variable t which controls all of x, y, and z. Of course, it is clear that z controls both x and y, so the simplest parameterization is to set $z = t$. Thus our solution may be written as

$$x = 2 - 3t, \qquad y = -1 - 2t, \qquad z = t.$$

Since we are free to choose z to be *any real number*, this system has infinitely many solutions. For example, setting $t = 1$ yields the solution

$$(-1, -3, 1),$$

while setting $t = -3$ yields a different solution

$$(11, 5, -3).$$

2. Upon inspecting the augmented matrix

$$\begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 0 & 0 & | & 1 \end{pmatrix},$$

it becomes clear that something strange is going on here. The augmented matrix corresponds to a system of two equations in three unknowns, say x, y, and z, but the second line of the equation implies that

$$0x + 0y + 0z = 1, \text{ or } 0 = 1,$$

which is simply untrue. Since there are *no* numbers x, y, and z so that $0x + 0y + 0z = 1$, so the system has no solution, i.e. is inconsistent.

3. The augmented matrix
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -3 \end{array}\right)$$
corresponds to a system of three equations in three unknowns; again, we will use x, y, and z. The equations are
$$\begin{aligned} x &= 4 \\ y &= \tfrac{1}{2} \\ z &= -3. \end{aligned}$$
Clearly there is a single (unique) solution to the system, namely
$$(4, \tfrac{1}{2}, -3).$$

Gauss-Jordan Elimination

It should be clear by now that it is quite straightforward to solve systems whose augmented matrices are in reduced-row echelon form. Thus our next task will be to learn an algorithm for finding the reduced row echelon form for any augmented matrix; the algorithm we discuss below is called *Gauss-Jordan elimination*, and will always rewrite a matrix in reduced row echelon form.

Before we proceed, we should make one quick note–while we may always use Gauss-Jordan elimination to put a matrix in reduced row echelon form, and thus to solve the associated system of equations, it is not the only solution algorithm. Other techniques such as back-substitution may be more convenient to use, depending on the form of the system. For sake of time, we only cover Gauss-Jordan elimination in this class.

The following steps constitute the method of Gauss-Jordan elimination:

1. Locate the first *column* of the augmented matrix that does not consist entirely of 0 entries.

2. If necessary, interchange the top row with another row so that the first entry in the column highlighted above is nonzero.

3. Create a leading 1 in the first row by multiplying that row by a suitable constant.

4. Add suitable multiples of the first row to the remaining rows to create 0s below the leading 1.

5. Ignore the current row and repeat steps $1 - 4$ with the remaining submatrix, until each row of the matrix either contains a leading 1 or consists only of 0 entries.

6. At this point, the matrix is in row echelon form, but *might not* be in reduced row echelon form. To put it in reduced row echelon form, add suitable multiples of the last nonzero row to each row above to create zeros above the leading 1 from this row; repeat the process with the remaining rows by working upwards.

Example. Use Gauss-Jordan elimination to put the matrix

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 4 \\ 2 & -3 & -1 & 6 & -7 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix}$$

in reduced row echelon form.

1. Locate the first *column* of the augmented matrix that does not consist entirely of 0 entries.

   The first such column is highlighted in red:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 4 \\ 2 & -3 & -1 & 6 & -7 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix}.$$

2. If necessary, interchange the top row with another row so that the first entry in the column highlighted above is nonzero.

   Notice that the top entry in our highlighted column is indeed 0, so we'll switch the first row, say with the second:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 4 \\ 2 & -3 & -1 & 6 & -7 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & -1 & 6 & -7 \\ 0 & 1 & 0 & -1 & 4 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix}.$$

3. Create a leading 1 in the first row by multiplying that row by a suitable constant.

   To get a leading 1 in the first row, we'll need to multiply by $\frac{1}{2}$:

$$\begin{pmatrix} 2 & -3 & -1 & 6 & -7 \\ 0 & 1 & 0 & -1 & 4 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix}.$$

4. Add or subtract suitable multiples of row 1 from the remaining rows below so that the leading 1 we just created is the *only* nonzero entry in its row.

   Inspecting our current matrix

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix},$$

we see that we only need to adjust the third row; to do so, let's add 3 times row 1 to row 3:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ -3 & 7 & 2 & -11 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ -3+3 & 7-\frac{9}{2} & 2-\frac{3}{2} & -11+9 & 14-\frac{21}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 & \frac{7}{2} \end{pmatrix}.$$

5. Ignore the first row and repeat steps 1 – 4 with the remaining submatrix.

   The entries of the first row are highlighted below to remind us to ignore them:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 & \frac{7}{2} \end{pmatrix}.$$

   We'll repeat steps 1 – 4 with the remaining (non-highlighted) rows.

   i. We need to locate the first column that has a nonzero entry–clearly the second column highlighted in red below:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 & \frac{7}{2} \end{pmatrix}.$$

   ii. Now the column highlighted in red above has a nonzero as its first entry, so there is no need to switch rows, and we may proceed to the next step.

   iii. Again ignoring the actual first row, notice that the current "first" row has a leading 1:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 & \frac{7}{2} \end{pmatrix}.$$

   Thus we do not need to adjust the entry, and we may proceed to the next step.

   iv. We need to create 0 entries below the leading 1 highlighted in red above; to do so, we subtract $\frac{5}{2}$ of the second row from the third:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 & \frac{7}{2} \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & \frac{5}{2}-\frac{5}{2} & \frac{1}{2}-0 & -2+\frac{5}{2} & \frac{7}{2}-\frac{20}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{13}{2} \end{pmatrix}.$$

Now that the second row also has a leading 1, we may ignore it as well and apply steps $1-4$ to the submatrix remaining when we delete rows 1 and 2, indicated below:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{13}{2} \end{pmatrix}.$$

i. The first column with a nonzero entry is highlighted in red below:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{13}{2} \end{pmatrix}.$$

ii. Since the first row (i.e., the only row) of the column highlighted above has a nonzero entry, we do not need to switch rows.

iii. The first nonzero entry of the row highlighted above is $\frac{1}{2}$, which we must adjust by 2 in order to get a leading 1:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{13}{2} \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

iv. There are no remaining rows, thus no more adjustments needed.

6. At this point, the matrix is in row echelon form, but *might not* be in reduced row echelon form. To put it in reduced row echelon form, add suitable multiples of the last nonzero row to each row above to create zeros above the leading 1 from this row; repeat the process with the remaining rows by working upwards.

   Notice that our matrix,

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix},$$

   is *not* in reduced row echelon form. Let's adjust to reduced row echelon form, starting with the last row, whose leading 1 is highlighted below:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

   We don't need to adjust the second row, but we will need to adjust the first one since it contains a nonzero entry above the leading 1 from the last row:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

We'll adjust the first row by adding $\frac{1}{2}$ the third row to it:

$$\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix} \rightarrow \begin{pmatrix} 1+0 & -\frac{3}{2}+0 & -\frac{1}{2}+\frac{1}{2} & 3+\frac{1}{2} & -\frac{7}{2}-\frac{13}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\frac{3}{2} & 0 & \frac{7}{2} & -10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

Now that the leading 1 from the third row has only 0 entries above it, we proceed to the second row, whose leading 1 and corresponding column are highlighted in red:

$$\begin{pmatrix} 1 & -\frac{3}{2} & 0 & \frac{7}{2} & -10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

Since there is a nonzero entry above the leading 1 from row 2, we will again need to adjust–this time by adding $\frac{3}{2}$ of row 2 to row 1:

$$\begin{pmatrix} 1 & -\frac{3}{2} & 0 & -\frac{7}{2} & -10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix} \rightarrow \begin{pmatrix} 1+0 & -\frac{3}{2}+\frac{3}{2} & 0+0 & \frac{7}{2}-\frac{3}{2} & -10+\frac{12}{2} \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{pmatrix}$$

is in reduced row echelon form.

Example. Solve the system of equations

$$\begin{aligned} x - z &= 4 \\ 2w - 3x - y + 6z &= -7 \\ -3w + 7x + 2y - 11z &= 14. \end{aligned}$$

The system above has augmented equation

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 4 \\ 2 & -3 & -1 & 6 & -7 \\ -3 & 7 & 2 & -11 & 14 \end{array}\right).$$

Of course, we've already found the reduced row echelon form of this matrix:

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 4 \\ 2 & -3 & -1 & 6 & -7 \\ -3 & 7 & 2 & -11 & 14 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -13 \end{array}\right)$$

The reduced row echelon form of the matrix is quite easy to solve: since w, x, and y correspond to leading 1s, and z does not, z is the only free variable. Our system can be written as

$$\begin{aligned} w + 2z &= -4 \\ x - z &= 4 \\ y + z &= -13. \end{aligned}$$

or

$$\begin{aligned} w &= -4 - 2z \\ x &= 4 + z \\ y &= -13 - z, \end{aligned}$$

Parameterizing $z = t$, we rewrite the solution as

$$w = -4 - 2t,\ x = 4 + t,\ y = -13 - t,\ \text{and } z = t.$$

Homogeneous Systems

Certain systems of equations have special properties that will be useful going forward. One such system is described below:

Definition. A system of equations is *homogeneous* if all of its constant terms are zero.

For example, the system

$$\begin{aligned} x + 3y &= 0 \\ -x + 2y - z &= 0 \\ 2x - y + z &= 0 \end{aligned}$$

is homogeneous; all of the terms that do not include a variable are 0s.

Homogeneous systems are special for an interesting reason–they always have *at least one* solution, the *trivial* solution, which we get by setting each variable equal to 0.

Using the previous example, it is clear that setting

$$x = 0,\ y = 0,\ \text{and } z = 0$$

yields a solution.

Since every homogeneous system has the trivial solution as a solution, the list of solution types for such a system is a bit shorter than in the general case–homogeneous systems are never inconsistent.

Key Point. Every homogeneous system of equations has either

1. a single (trivial) solution, or
2. infinitely many solutions.

Example. Use Gauss-Jordan elimination to solve the homogeneous system

$$\begin{aligned} 4x_1 - 12x_4 + 56x_5 &= 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{4}x_3 - \tfrac{3}{2}x_4 + \tfrac{15}{2}x_5 &= 0 \\ -2x_1 + x_2 - x_3 + 4x_4 - 26x_5 &= 0. \end{aligned}$$

The augmented matrix for the system is

$$\left(\begin{array}{ccccc|c} 4 & 0 & 0 & -12 & 56 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & -\frac{3}{2} & \frac{15}{2} & 0 \\ -2 & 1 & -1 & 4 & -26 & 0 \end{array}\right).$$

Let's apply Gauss-Jordan elimination to rewrite the system in reduced row echelon form: The first column with nonzero entries is highlighted in red below:

$$\left(\begin{array}{ccccc|c} \color{red}{4} & 0 & 0 & -12 & 56 & 0 \\ \color{red}{\frac{1}{2}} & 0 & \frac{1}{4} & -\frac{3}{2} & \frac{15}{2} & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right).$$

The first row of this column begins with a nonzero entry, so we need to rewrite the entire first row with a leading 1; multiplying the row by $\frac{1}{4}$ will accomplish this:

$$\left(\begin{array}{ccccc|c} \color{red}{4} & 0 & 0 & -12 & 56 & 0 \\ \color{red}{\frac{1}{2}} & 0 & \frac{1}{4} & -\frac{3}{2} & \frac{15}{2} & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & 0 \\ \color{red}{\frac{1}{2}} & 0 & \frac{1}{4} & -\frac{3}{2} & \frac{15}{2} & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right).$$

Next, we need to add multiples of the first row to the others so that the leading 1 we created above has only 0 entries below. We accomplish this with the second row by subtracting $\frac{1}{2}$ of row 1 from row 2:

$$\left(\begin{array}{ccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & 0 \\ \color{red}{\frac{1}{2}} & 0 & \frac{1}{4} & -\frac{3}{2} & \frac{15}{2} & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & 0 \\ \color{red}{\frac{1}{2}-\frac{1}{2}} & 0-0 & \frac{1}{4}-0 & -\frac{3}{2}+\frac{3}{2} & \frac{15}{2}-7 & 0+0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & 0 \\ \color{red}{0} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & 0 \end{array}\right)$$

.

Next, we need to alter the third row so that its first entry is a 0; we can do so by adding 2 times the first row to row 3:

$$\left(\begin{array}{cccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & & 0 \\ \color{red}{0} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & & 0 \\ \color{red}{-2} & 1 & -1 & 4 & -26 & & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & & 0 \\ \color{red}{0} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & & 0 \\ \color{red}{-2+2} & 1+0 & -1+0 & 4-6 & -26+28 & & 0+0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} \color{red}{1} & 0 & 0 & -3 & 14 & 0 \\ \color{red}{0} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ \color{red}{0} & 1 & -1 & -2 & 2 & 0 \end{array}\right).$$

Now that the first column has 0s below its leading 1, we may "forget" the row containing the leading 1 and repeat the process with the remaining submatrix. We highlight row 1 in blue to remind us that we are ignoring it:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \end{array}\right). \tag{1}$$

To proceed, we look for the next column that has a nonzero entry; in this case, it's the second, highlighted in red below:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & \color{red}{0} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \color{red}{1} & -1 & -2 & 2 & 0 \end{array}\right). \tag{2}$$

Since our new first row has a zero entry in the column highlighted in red, we begin by switching the two remaining rows to create a leading 1 in the highlighted column:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & \color{red}{0} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \color{red}{1} & -1 & -2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & \color{red}{1} & -1 & -2 & 2 & 0 \\ 0 & \color{red}{0} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \end{array}\right).$$

Next, we need to create 0s in all entries below the leading 1; fortunately, the leading 1 we just created already has 0s below it.

Thus we move to the next step: ignore rows 1 and 2 and focus on the remaining submatrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \end{array}\right).$$

The first column of this submatrix with a nonzero entry is the 3rd, highlighted in red below:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & \color{red}{\frac{1}{4}} & 0 & \frac{1}{2} & 0 \end{array}\right).$$

We need to adjust this row to create a leading 1, which we accomplish by multiplying row 3 by 4:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & \color{red}{\frac{1}{4}} & 0 & \frac{1}{2} & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 2 & 0 \end{array}\right).$$

The matrix above is in row echelon form, *but not* reduced row echelon form, so we continue to apply row operations to it. We work backwards from row 3, and create 0 entries above each leading 1. The third row is highlighted in red below:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right).$$

Notice that the only nonzero entry above the leading 1 from row 3 occurs in the second row; thus we add row 3 to row 2 to eliminate this nonzero entry:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0+0 & 1+0 & -1+1 & -2+0 & 2+2 & 0+0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right).$$

Notice that the last matrix above,

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & 14 & 0 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right),$$

is actually in reduced row echelon form. We have reduced the matrix as far as possible, and we may now easily find the solution to the associated linear system. The augmented matrix above indicates that

$$\begin{aligned} x_1 - 3x_4 + 14x_5 &= 0 \\ x_2 - 2x_4 + 4x_5 &= 0 \\ x_3 + 2x_5 &= 0. \end{aligned}$$

Since x_4 and x_5 are the only variables that do not correspond to leading 1s, they are both *free*, and we will parameterize them, say as

$$x_4 = s \text{ and } x_5 = t.$$

We write the final solution to the system as

$$x_1 = 3s - 14t,\ x_2 = 2s - 4t,\ x_3 = -2t,\ x_4 = s,\ \text{and } x_5 = t.$$

We saw earlier that every homogeneous system of equations–that is, every system whose constant terms are all 0–has either

1. a single (trivial) solution, or

2. infinitely many solutions.

It turns out that it is quite easy to distinguish between the two cases above by using the following theorem:

Theorem. A homogeneous system with more unknowns than equations has infinitely many solutions.

We can illustrate the theorem using the previous example; we saw that the system

$$\begin{aligned} 4x_1 - 12x_4 + 56x_5 &= 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{4}x_3 - \tfrac{3}{2}x_4 + \tfrac{15}{2}x_5 &= 0 \\ -2x_1 + x_2 - x_3 + 4x_4 - 26x_5 &= 0 \end{aligned}$$

(in 3 equations with 5 unknowns) has solution

$$x_1 = 3s - 14t,\ x_2 = 2s - 4t,\ x_3 = -2t,\ x_4 = s,\ \text{and } x_5 = t.$$

The theorem indicates that the system should have infinitely many solutions, which we quickly verify by noting that we are free to choose the parameters s and t in any way we like.