Determinants

In section 4, we discussed inverses of matrices, and in particular asked an important question:

How can we tell whether or not a particular square matrix A has an inverse?

We will be able to answer this question soon–but first, we must take a few moments to discuss a property of square matrices called the *determinant*.

You are probably already familiar with calculating determinants for 2×2 and 3×3 matrices; unfortunately, there is no simple formula for calculating determinants of larger matrices. Accordingly, we will in this section develop a method for finding the determinant of *any* square matrix.

Determinants of 2×2 Matrices

The determinant function is quite simple to use if the matrix in question is 2×2:

Definition. The determinant of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

denoted by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ or } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is the number

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

As a quick example, let's calculate the determinant of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 9 & -6 \end{pmatrix}.$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & -2 \\ 9 & -6 \end{vmatrix} \\ &= 3 \cdot (-6) - (-2) \cdot 9 \\ &= -18 + 18 \\ &= 0. \end{aligned}$$

Notice that the determinant function sent the *matrix* A to the *number* 0.

Key Point. The determinant function is a function that sends square matrices to numbers; if we let $\mathcal{M}_n$ be the set of all $n \times n$ matrices with real entries, we can write

$$\det : \mathcal{M}_n \to \mathbb{C}$$

to indicate the correspondence. If all of the entries of the matrix A are real, then $\det A$ will be a purely real number; otherwise, $\det A$ might have an imaginary component.

Of course, we want to be able to calculate the determinant of any square matrix, not just the 2×2 variety; there are actually many alternate methods for doing so (in fact, you are probably quite familiar with the "diagonal" method for 3×3 matrices). All of the methods will produce equivalent results, but the method that we will learn in this section requires us to use the definition of the determinant for 2×2 matrices.

Minors and Cofactors

The method that we are going to learn for calculating determinants is called *cofactor expansion*. Accordingly, we need to define cofactors, and the closely related concept of *minors*.

To fully understand minors and cofactors, we need to be comfortable with the idea of a *submatrix*, which is simply any matrix living inside a larger one. As an example, let's think about the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}.$$

We can create a submatrix of A by, say deleting its first row and first column:

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix} \to A_{11} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}.$$

We use the notation A_{11} to indicate that it is the submatrix of A that results from deleting row 1 and column 1 of A.

Similarly, we can get another submatrix A_{12} of A by deleting, say, the first row and second column:

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix} \to A_{12} = \begin{pmatrix} -2 & 4 \\ 2 & 1 \end{pmatrix}.$$

We are now ready for the relevant definitions:

Definition. Let A be a square matrix with entries a_{ij}. The *minor* M_{ij} of entry a_{ij} is the determinant of the submatrix A_{ij} of A that results from deleting row i and column j.

The *cofactor* C_{ij} of entry a_{ij} is the quantity

$$C_{ij} = (-1)^{i+j} M_{ij},$$

that is

$$C_{ij} = \begin{cases} M_{ij} & i+j \text{ is even} \\ -M_{ij} & i+j \text{ is odd.} \end{cases}$$

Example

Calculate the minors and cofactors of entries a_{11} and a_{12} of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}.$$

In order to calculate the minor M_{11} of entry a_{11}, we need to find the submatrix A_{11} that results from A by deleting row 1 and column 1; fortunately, we already calculated above that

$$A_{11} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}.$$

According to the definition, M_{11} is the determinant of this submatrix, i.e.

$$\begin{aligned} M_{11} &= \det A_{11} \\ &= \det \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \\ &= 0 \cdot 1 - 4 \cdot 1 \\ &= -4. \end{aligned}$$

So $M_{11} = -4$, and since

$$C_{11} = (-1)^{1+1} M_{11} = 1 \cdot -4,$$

we know that $C_{11} = -4$ as well.

Let's calculate M_{12}: again, we need to start with the submatrix A_{12} that results from A by deleting row 1 and column 2:

$$A_{12} = \begin{pmatrix} -2 & 4 \\ 2 & 1 \end{pmatrix}.$$

Since $M_{12} = \det A_{12}$, we calculate

$$\begin{aligned} M_{12} &= \det A_{12} \\ &= \det \begin{pmatrix} -2 & 4 \\ 2 & 1 \end{pmatrix} \\ &= -2 \cdot 1 - 4 \cdot 2 \\ &= -10. \end{aligned}$$

Finally, to get the cofactor of a_{12}, we calculate

$$C_{12} = (-1)^{1+2} M_{12} = -1(-10) = 10.$$

Key Point. As mentioned in the definition, C_{ij} is always equal either to M_{ij} (if $i+j$ is even), or to $-M_{ij}$ (if $i+j$ is odd). While you can certainly use this formula to determine which sign is correct, it's often easier to make that choice by use a simple observation: the cofactor C_{11} of a_{11} is always positive, and the cofactor switches signs along any checkerboard pattern.

For example, if we wished to determine if the cofactor C_{43} of entry a_{43} in the matrix

$$\begin{pmatrix} 0 & 1 & 1 & -5 & -1 \\ -2 & 3 & 2 & 1 & 0 \\ 10 & 8 & 3 & 0 & 1 \\ -3 & 4 & 4 & 1 & 0 \\ 1 & 0 & 2 & 7 & 1 \end{pmatrix}$$

is equal to M_{ij} or to $-M_{ij}$, we can start by picking *any* checkerboard pattern (right, left, up, or down) from entry a_{11} to entry a_{43}, say as follows:

$$\begin{pmatrix} \color{red}{0} & 1 & 1 & -5 & -1 \\ \color{red}{-2} & \color{red}{3} & \color{red}{2} & \color{red}{1} & 0 \\ 10 & 8 & 3 & \color{red}{0} & 1 \\ -3 & 4 & \color{red}{4} & \color{red}{1} & 0 \\ 1 & 0 & 2 & 7 & 1 \end{pmatrix}.$$

Now we know that C_{11} keeps the sign of M_{11}; to determine what happens with C_{43}, follow the checkerboard pattern from a_{11} to a_{43}, switching signs at each step:

$$\begin{pmatrix} \color{red}{+} & 1 & 1 & -5 & -1 \\ \color{red}{-} & \color{red}{+} & \color{red}{-} & \color{red}{+} & 0 \\ 10 & 8 & 3 & \color{red}{-} & 1 \\ -3 & 4 & \color{red}{-} & \color{red}{+} & 0 \\ 1 & 0 & 2 & 7 & 1 \end{pmatrix}.$$

Since we ended at a $-$ sign, we know that $C_{43} = -M_{43}$.

Calculating Determinants Using Cofactor Expansion

We are almost ready to record a method for calculating the determinant of any square matrix. We need a bit more data to be able to do so, starting with the definition of *cofactor expansion*:

Definition. The *cofactor expansion* of an $n \times n$ matrix A along row i is the sum of the products of the entries a_{ik} of the ith row with their cofactors C_{ik}, that is

$$\text{cofactor expansion of } A \text{ along row } i = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2} + C_{i2} + \ldots + a_{in}C_{in}.$$

The cofactor expansion of A along row j is the sum of the products of the entries a_{kj} of the jth column with their cofactors C_{kj}, that is

$$\text{cofactor expansion of } A \text{ along column } j = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j} + C_{2j} + \ldots + a_{nj}C_{nj}.$$

Example

Calculate the cofactor expansion of

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}$$

along

1. the first row, and
2. the second column.

1. The formula for the cofactor expansion of A along the first row is

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13};$$

fortunately, we already have most of the data we need to make this calculation. We have already determined that

$$\begin{array}{lll} C_{11} = -4 & C_{12} = 10 & \\ a_{11} = 1 & a_{12} = 3 & a_{13} = 1. \end{array}$$

To make the calculation, we simply need to find C_{13}. Let's start with the minor M_{13}, the determinant of the matrix obtained from A by deleting its first row and third column:

$$A_{13} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}.$$

Now

$$\begin{aligned} M_{13} &= \det A_{13} \\ &= \det \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix} \\ &= -2 \cdot 1 - 0 \cdot 2 \\ &= -2, \end{aligned}$$

and

$$C_{13} = (-1)^{1+3} M_{13} = 1 \cdot -2 = -2,$$

so our chart is completed:

$$\begin{array}{lll} C_{11} = -4 & C_{12} = 10 & C_{13} = -2 \\ a_{11} = 1 & a_{12} = 3 & a_{13} = 1. \end{array}$$

Using the formula above for the cofactor expansion along row 1 of A, we have

$$\begin{aligned} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} &= 1 \cdot -4 + 3 \cdot 10 + 1 \cdot -2 \\ &= -4 + 30 - 2 \\ &= 24. \end{aligned}$$

2. Next, let's find the cofactor expansion of A along the second column. The formula for the expansion is

$$a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32};$$

we already have much of the necessary data:

We know that

$$\begin{array}{ccc} C_{12} = 10 & C_{22} = ? & C_{32} = ? \\ a_{12} = 3 & a_{22} = 0 & a_{32} = 1. \end{array}$$

Thus we simply need to calculate the cofactors C_{22} and C_{32}. However, upon closer inspection, it's clear that we don't even need to find C_{22}! When we calculate the cofactor expansion along column 2, C_{22} shows up only once, as a factor in the product $a_{22}C_{22}$. Since $a_{22} = 0$, we already know that $a_{22}C_{22} = 0$ as well.

So we simply need to calculate C_{32}, again starting with the submatrix A_{32} obtained from A by deleting row 3 and column 2:

$$A_{32} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Now

$$\begin{aligned} M_{32} &= \det A_{32} \\ &= \det \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \\ &= 1 \cdot 4 - 1 \cdot (-2) \\ &= 6, \end{aligned}$$

and

$$C_{32} = (-1)^{3+2} M_{32} = -1 \cdot 6 = -6,$$

and we have:

$$\begin{array}{ccc} C_{12} = 10 & C_{22} = ? & C_{32} = 6 \\ a_{12} = 3 & a_{22} = 0 & a_{32} = 1. \end{array}$$

Using the formula above for the cofactor expansion along column 2 of A, we have

$$\begin{aligned} a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} &= 3 \cdot 10 + 0 + 1 \cdot -6 \\ &= 30 - 6 \\ &= 24. \end{aligned}$$

Determinants

You may have noticed something quite interesting about the last example–the cofactor expansion of A along row 1 was exactly the same number as the cofactor expansion of A along column 2. If you think that this is too remarkable to be a fluke, you're absolutely right, as indicated by the following theorem:

Theorem. Given an $n \times n$ matrix A, its cofactor expansion along *any* row or *any* column is the same.

The proof of this theorem is beyond the scope of this course; however, its consequences are quite practical. In terms of the example above, we could have calculated the expansion of A along row 2, row 3, column 1, or column 3; regardless, we would have gotten 24 as our answer each time.

We are finally ready to define the determinant function for any square matrix:

Definition. Let A be an $n \times n$ matrix. The *determinant* of A, denoted $\det A$, is the cofactor expansion of A along any row or column.

With this definition in mind, we should note that we have already calculated the determinant of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix};$$

since its cofactor expansion is 24, we have

$$\det A = 24.$$

As you may have guessed, cofactor expansion can be a rather tedious algorithm. However, there are occasions in which the method is quite easy to use–the key is to make smart choices about which rows or columns to choose in the expansion.

As indicated by the theorem, *any time* that we need to calculate a cofactor expansion, we are free to choose whichever column or row we would like to use in the expansion (the answer will always be the same!). This is a fact that we can exploit in our computation.

We have actually already seen an example of the way in which the expansion becomes easier: when we discussed the cofactor expansion of

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}$$

along its second column, we found that the expansion was fairly simple, since *the second entry of the column is* 0. Because of this, we didn't have to go to the trouble of calculating C_{22}, which saved us a good bit of time.

The following example illustrates the way in which expanding wisely (i.e., expanding rows or columns with lots of 0s) can save us a great deal of time in making the determinant calculation.

Example

Given

$$A = \begin{pmatrix} 3 & 0 & -2 & 0 \\ 1 & 1 & 5 & 1 \\ 2i & 0 & 0 & 0 \\ -1 & 3 & 2 & 7 \end{pmatrix},$$

find $\det A$.

Since A is a 4×4 matrix, we have a good bit of calculation to do in order to find its determinant. Since the determinant is just the cofactor expansion of A along *any* row or column, we should try to calculate the expansion in a way that will minimize arithmetic.

Upon inspecting the matrix, you may have noticed that the third row of A has three 0s–more than any of the other columns or rows. Because of this, we should choose this row for the expansion. Thus our formula is

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}.$$

However, since each of a_{32}, a_{33}, and a_{34} is 0, the formula reduces to

$$\det A = a_{31}C_{31}.$$

Our next step is to calculate C_{31}, so we need the submatrix A_{13} of A obtained by deleting row 3 and column 1:

$$A = \begin{pmatrix} \color{red}{3} & 0 & -2 & 0 \\ \color{red}{1} & 1 & 5 & 1 \\ \color{red}{2i} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ \color{red}{-1} & 3 & 2 & 7 \end{pmatrix} \to A_{31} = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 5 & 1 \\ 3 & 2 & 7 \end{pmatrix}.$$

The minor M_{31} is the determinant of A_{31}, i.e.

$$M_{31} = \det \begin{pmatrix} 0 & -2 & 0 \\ 1 & 5 & 1 \\ 3 & 2 & 7 \end{pmatrix}.$$

Of course, to calculate $\det A_{31}$, we will need to go through another iteration of cofactor expansion. Again, we should try to choose a row or column for the expansion that will minimize the amount of work we have to do. Inspecting A_{31}, it is clear that the first row should be our choice for the expansion; it contains two 0s, more than any other row or column.

Using b_{ij}s to indicate entries of A_{31} and D_{ij}s to indicate the corresponding cofactors, the formula for the determinant of A_{31} reduces from

$$\det A_{31} = b_{11}D_{11} + b_{12}D_{12} + b_{13}D_{13}$$

to

$$\det A_{31} = b_{12}D_{12}.$$

To get the cofactor D_{12} of A_{31}, we again start by finding the determinant of the submatrix

$$\begin{pmatrix} 1 & 1 \\ 3 & 7 \end{pmatrix}.$$

Since

$$\begin{aligned} \det \begin{pmatrix} 1 & 1 \\ 3 & 7 \end{pmatrix} &= 1 \cdot 7 - 1 \cdot 3 \\ &= 4, \end{aligned}$$

the desired cofactor D_{12} is

$$D_{12} = (-1)^{1+2} \cdot 4 = -4.$$

Thus the determinant of A_{31} is

$$\begin{aligned} \det A_{31} &= b_{12}D_{12} \\ &= -2 \cdot -4 \\ &= 8. \end{aligned}$$

Of course, we calculated $\det A_{31}$ to get the minor M_{31}; so

$$M_{31} = \det A_{31} = 8,$$

and the corresponding cofactor is

$$C_{31} = (-1)^{3+1} M_{31} = 8.$$

Since our formula for the determinant of A was

$$\det A = a_{31}C_{31},$$

and we know that $a_{31} = 2i$ and $C_{31} = 8$, we have

$$\det A = 16i.$$

Determinants of Triangular and Diagonal Matrices

The idea presented above for choosing rows and columns wisely in the cofactor expansion leads to a simple yet powerful observation about determinants of triangular and diagonal matrices:

Theorem. The determinant of a diagonal, upper triangular, or lower triangular matrix is the product of its diagonal entries.

Let's quickly think about why the theorem works, using the upper triangular matrix

$$A = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

In order to find the determinant of A, it is clear that we should expand along the first column, since it contains mostly 0s. Thus

$$\det A = 3 \cdot \det \begin{pmatrix} 2 & -3 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

To find the determinant in the line above, we again need a cofactor expansion, and once again it is clear that we should use the first column for the calculation: so

$$\begin{aligned} \det A &= 3 \cdot \det \begin{pmatrix} 2 & -3 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \\ &= 3 \cdot 2 \cdot \det \begin{pmatrix} -1 & 1 \\ 0 & 4 \end{pmatrix}. \end{aligned}$$

Of course, this last determinant is easy to calculate, particularly so since its $2, 1$ entry is 0; we have

$$\begin{aligned} \det A &= 3 \cdot \det \begin{pmatrix} 2 & -3 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \\ &= 3 \cdot 2 \cdot \det \begin{pmatrix} -1 & 1 \\ 0 & 4 \end{pmatrix} \\ &= 3 \cdot 2 \cdot (-1 \cdot 4 - 0 \cdot 1) \\ &= 3 \cdot 2 \cdot (-1) \cdot 4 \\ &= -24, \end{aligned}$$

which is just the product of the diagonal entries.